A STUDY OF CERTAIN SUBCLASSES OF UNIVALENT ANALYTIC FUNCTIONS

by

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CERTIFICATE

It is certified that the work contained in the thesis entitled "**A STUDY OF CERTAIN SUBCLASSES OF UNIVALENT ANALYTIC FUNCTIONS**", by **Vedanabhatla Srinivas**, has been carried out under our supervision and that this work has not been submitted elsewhere for a degree.

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SYNOPSIS

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The thesis consists of seven chapters.

Chapter I is introduction and consists of basic definitions and known results used in the subsequent chapters of the thesis.

Chapter II is devoted to the study of the class $CVG(R_1,R_2)$, $0 \le R_1 \le R_2 \le \infty$, $R_2 \ge 1$ recently introduced by D. Styer and D.J. Wright [Proc. Amer. Math. Soc., **109**, No. 4, (1990), 981-990]. First, certain necessary conditions in terms of $d^* = \sup_{\zeta \in \partial U} |f(\zeta)|$ for f to be in $CVG(R_1,R_2)$ are determined in this chapter. One of these conditions gives a lower bound on d^* if $f \in CVG(R_1,R_2)$. Necessary and sufficient conditions are determined for R_1 to be equal to R_2 if $f \in CVG(R_1,R_2)$. Further in this chapter, the results of following nature are obtained for the class $CVG(R_1,R_2)$:

(i) growth bounds on $|f(z)|$ in terms of d^* (ii) lower growth bound on $|f(z)|$ involving $|z|$ and R_1 in the disc $|z| < R_0$, $R_0 \cong 0.543$ (iii) bounds on the functional $|(f(z_1) - f(z_2))/(z_1-z_2)|$ for certain distinct z_1, z_2 in the unit disc U (iv) distortion bounds on $|f'(z)|$ in the disc $|z| \leq 3 - \sqrt{8}$ (v) a rotation theorem when $R_2 < \infty$ and (vi) an estimate of Euclidean curvature $k(f;z)$. Finally, an open problem of A.W. Goodman [Proc. Amer. Math. Soc. **92**, NO. 4, (1984), 541-546] is solved for $CVG(R_1,R_2)$, $R_2 < \infty$ in this chapter by showing that $CVG(R_1,R_2) \subseteq CV(\gamma)$ where $\gamma = 2R_2 - 1 - 2\sqrt{R_2^2 - R_2}$ and γ is the largest possible such a number.

In Chapter III the concept of α-curvature, $\alpha < 1$ is introduced for functions analytic and locally univalent in the unit disc U and the resulting classes $CV_\alpha(R_1,R_2)$, $0 \leq R_1 \leq R_2 \leq \infty$, $0 \leq \alpha < 1$ and $C_\alpha(K)$, $K > 0$, $\alpha < 1$ are studied. For $0<R_1 \leq R_2 < \infty$, functions in $CV_\alpha(R_1,R_2)$ are called convex functions of bounded α-type. While some of the results contained in the chapter for the classes $CV_\alpha(R_1,R_2)$ and $C_\alpha(K)$ are analogues of the results of A.W. Goodman [op. cit.; ibid. **97**, No. 2, (1986), 303-306] for the class $CV(R_1,R_2) \equiv CV_0(R_1,R_2)$ and those of K.-J.Wirths [Proc. Amer. Math. Soc. **103**, No. 2 (1988) 525-530; Ann. Univ. Mariae Curie-Sklodowska Sect. A **41** (1987) 153-158 (1989)] for the class $C(K) \equiv C_0(K)$, several other results found in the chapter are new. First, the sharp lower bound on γ so that a function in $CV_\alpha(R_1,R_2)$, $R_2 < \infty$ is convex of order γ is obtained. Then an

integral operator that transforms convex functions of bounded α-type into convex functions of bounded type [Goodman (1984)] is studied. This integral operator is helpful in finding distortion bounds for the class $CV_\alpha(R_1,R_2)$. The other results obtained in this chapter for the class $CV_\alpha(R_1,R_2)$ are of the following nature: (i) growth theorem on $|f(z)|$ (ii) bounds on the functional $|(f(z_1) - f(z_2))/(z_1-z_2)|$ for certain distinct z_1, z_2 in U and (iii) a rotation theorem when $R_2 < \infty$. Next, necessary and sufficient conditions for a function f to be in the class $C_\alpha(K)$ are found. The other results for the class $C_\alpha(K)$ found in this chapter include (i) the bounds on the second and third Taylor series coefficients for functions in $C_\alpha(K)$ and (ii) distortion bounds.

A function f analytic in the unit disc U, normalized by $f(0) = 0$, $f'(0) = 1$ with $f(z) \neq 0$ in the punctured disc $U\backslash\{0\}$, may be expressed as (1) $f(z) = \psi(g) = z/g(z)$ in U, where (2) $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ in U. For $\psi(g)$ varying over a certain class of polynomials of degree atmost n ($n \geq 2$), the reciprocal coefficient region is defined in Chapter IV. First, the reciprocal coefficient regions of certain classes of univalent polynomials are determined in this chapter. Further necessary and sufficient conditions in terms of the reciprocal coefficient regions are found for a polynomial $\psi(g)$ to be in certain subclasses of starlike functions or convex functions or in certain other known subclasses of univalent functions. The reciprocal

coefficient regions of certain classes of univalent polynomials determined in this work are improvements over the reciprocal coefficient region found by H.Silverman and E.M. Silvia [Complex Variables Theory and applns. 5 (1986), 313-321] for certain bigger class of univalent polynomials.

In Chapter V, necessary and sufficient conditions in terms of the coefficients b_n are determined for the function $\psi(g)$ to be in certain classes of analytic functions. Necessary and sufficient conditions in terms of the reciprocal coefficient regions for a trinomial $\psi(g)$ to be in certain subclasses of univalent functions with univalent Gelfond-Leontev derivatives in U are determined. The bounds on b_n, $1 \leq n \leq 4$ are found, when a trinomial $\psi(g)$ is univalent and has univalent Gelfond-Leontev derivative in U. Necessary as well as sufficient conditions in terms of $\{b_n\}_{n=1}^{\infty}$ are determined for the function $\psi(g)$ to be in each one of certain subclasses of starlike functions or spirallike functions. Some of the results in this chapter generalize the results of M.O. Reade; H. Silverman and P. G. Todorov [Rend. Circ. Mat. Palermo (2) 33 No. 2 (1984), 265-272].

In Chapter VI the support points, growth theorems and distortion theorems for certain new subclasses of analytic functions are determined. Among the new classes considered are (i) the class $A\{n,M_k\}$, $n = 1,2,3,\ldots$, consisting of functions $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ in U with $\arg a_k = -\arg M_k$ for $a_k \neq 0$ and

$\sum_{k=n+1}^{\infty} M_k a_k \leq 1$ where $\{M_k\}_{k=n+1}^{\infty}$ is a sequence of nonzero complex numbers and (ii) the class $A_o(n,B_k,z_o)$, $n = 1,2,3,\ldots$, z_o real, $0 < |z_o| < 1$, $B_k > 0$, consisting of functions with two fixed points 0, z_o. The extreme points of each of the classes $A\{n,M_k\}$ and $A_o(n,B_k,z_o)$ are determined. The support points of certain subclasses of univalent functions with univalent Gelfond-Leontev derivatives in U and the class $A\{n,M_k\}$ with $|M_k| \geq k$ are described. The support points of the class $A_o(n,B_k,z_o)$ with $B_k \geq k$ are also determined. Growth and distortion properties for functions in the classes $A\{n,M_k\}$ and $A_o(n,B_k,z_o)$ are also found in this chapter. Finally, the radii for starlikeness, convexity etc., in the class $A\{n,M_k\}$ are determined.

In Chapter VII, the positivity of real parts of linear combinations of analytic functions is studied. Let $\phi = \phi(f,f',f'')$ and $\psi = \psi(f,f',f'')$, where f is in either of the classes of (i) starlike functions (ii) convex functions (iii) starlike functions of order 1/2 (iv) prestarlike functions of order α, be such that ϕ and ψ as functions of z are analytic and Re $\phi > 0$ in the domain under consideration. In this chapter, the following problems with special choices of functionals ϕ and ψ are studied : (i) To find the largest ρ, $0 < \rho < 1$, such that, Re $(\phi + \psi) > 0$ in the disc $|z| < \rho$ (ii) To find the ranges of scalars λ and μ such that $\mathrm{Re}(\lambda\phi + \mu\psi) > 0$ in the unit disc U. Finally, sufficient conditions regarding Problem (ii) in terms of λ, μ and $\{b_n\}_{n=1}^{\infty}$ are determined where f and b_n's are as in (1) and (2) respectively.

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Srinivas

V. Srinivas

To
my parents

CONTENTS

CHAPTER I

INTRODUCTION

1.1 A complex valued function is said to be analytic in a domain Ω (a non-empty open connected subset of the complex plane $\mathbb{C}$) if it has a uniquely determined derivative at each point of Ω. A function f is said to be univalent in a domain Ω if it never takes any value more than once, that is, the condition $f(z_1) = f(z_2)$, $z_1, z_2 \in \Omega$, implies $z_1 = z_2$. A necessary condition for an analytic function f to be univalent in Ω is that $f'(z) \neq 0$ in Ω. That this condition is not sufficient can be seen by considering the function $f(z) = e^z$ whose derivative never vanishes but clearly it is not univalent in $\mathbb{C}$.

In the study of univalent functions, the Riemann mapping theorem plays an important role. The theorem states that if Ω is a simply connected domain which is a proper subset of the complex plane and z_0 is a given point in Ω, then there is a unique function f which maps Ω conformally onto the unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$ and has the properties $f(z_0) = 0$ and $f'(z_0) > 0$. Thus, for the study of geometric properties of functions univalent and analytic in a simply connected domain which is a

proper subset of the complex plane, one may therefore confine, without loss of generality, to functions univalent and analytic in the unit disc U.

If g is univalent and analytic in U, so is the function $f(z)=(g(z)-g(0))/g'(0)$ since $g'(0) \neq 0$. Thus it is enough to consider univalent analytic functions in U satisfying $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{A}$ denote the class of functions f analytic in U, $\mathcal{A}_1$ denote the subclass of functions $f \in \mathcal{A}$ normalized by the conditions $f(0) = 0$, $f'(0) = 1$ and let S be the subclass of functions f in $\mathcal{A}_1$ that are univalent in U. The Taylor series expansion of such a function f(z) about the origin has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in U. \tag{1.1.1}$$

The study of theory of univalent functions was initiated by Koebe [64] in 1907 in a paper on the uniformization of analytic curves. He proved that the ranges of all functions in S contain a common disc $|w| < b$, where b is an absolute constant. The Koebe function $k(z) = z/(1-z)^2$ shows that $b \leq 1/4$. Bieberbach [11] established that $b = 1/4$. He also proved in the same paper that if $f \in S$, than $|a_2| \leq 2$ with equality occurring only for the rotations $e^{-i\theta} k(e^{i\theta} z)$, θ real, of the Koebe function $k(z)$. Motivated by these extremal properties of the Koebe function, Bieberbach conjectured that for every $f \in S$ and of the form (1.1.1),

$$|a_n| \leq n, \qquad n = 2, 3, \ldots \tag{1.1.2}$$

Equality occurs in (1.1.2) for each n, if and only if, f is the Koebe function $k(z)$ or one of its rotations.

A conjecture stronger than Bieberbach conjecture is due to Robertson [111] which asserts that if $f \in S$, then the coefficients of odd univalent functions

$$h(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \ldots$$

satisfy the inequality

$$\sum_{k=1}^{n} |c_{2k-1}|^2 \leq n, \qquad n = 1, 2, \ldots$$

where $c_1 = 1$.

In 1971, Milin [80] proposed the following conjecture: If $f \in S$, then

$$\sum_{m=1}^{n} \sum_{k=1}^{m} (k|\gamma_k|^2 - 1/k) \leq 0, \qquad n = 1, 2, \ldots$$

where γ_n, $n = 1, 2, \ldots$ are given by

$$\log (f(z)/z) = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$

Milin conjecture is the strongest in the sense that it implies Robertson conjecture and hence Bieberbach conjecture.

Recently, Milin conjecture has been proved by Louis de Branges [24]. With this, both Bieberbach conjecture and Robertson conjecture stand proved in affirmative. A different and simplified version of Louis de Brange's proof has now been given by Fitzgerald and Pommerenke [30]. Yet, the univalent

function theory has got interesting open problems. For instance, Ruscheweyh [117] has conjectured the following:

Let $D = \{g \in A_1 : |g''(z)| \leq \text{Re } g'(z),\ z \in U\}$ and S^* be as in Definition 1.2.1.

<u>Conjecture</u>: Let f be in the closed convex hull of S, $g \in D$. Then $f*g \in S^*$ where $f*g$ denotes the convolution defined in Section 1.3. This conjecture is verified partially in [117] and is stronger than the Bieberbach conjecture. This still remains open for investigation.

Some more open problems in the univalent function theory can be found in Schober [123], Shaffer [126], Ahuja [1], Hayman [53], Singh [144], Barnard [7], Brannan and Hayman [18] etc..

During the process of solution of the Bieberbach and related conjectures, several subclasses of univalent functions, important in their own right,were introduced and different techniques like Loewner parametric method, convolution techniques, variational methods, subordination technique etc. were discovered. All these developments are amply reflected in Bernardi's Bibliography of Univalent Functions [9]. The texts of Nehari [89], Goluzin [39], Jenkins [58], Pommerenke [102], Schober [122], Duren [26] and Goodman [42],[43] cover almost all the fundamental aspects of the theory of univalent functions.

1.2 For a function $f \in S$, Nevanlinna [90] found the bounds on the growth of $|f(z)|$. Thus for $z \in U$,

$$(1.2.1) \qquad \frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}, \qquad |z| = r.$$

Bounds on $|f(z)|$ are called growth bounds.

The information about local dilation factor of a conformal mapping $f(z)$ is obtained, by finding bounds on $|f'(z)|$. Such bounds on $|f'(z)|$ are called distortion bounds. Nevanlinna proved [90] that, if $f \in S$ and $z \in U$, then

$$(1.2.2) \qquad \frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}, \qquad |z| = r.$$

In both the inequalities (1.2.1) and (1.2.2), only the Koebe function $k(z) = z/(1-z)^2$ and its rotations give sharpness. Grownwall [47] obtained an upper bound of $|f'(z)|$ in terms of the area of the image domain $f(U)$ and r. Thus, he found

$$(1.2.3) \qquad |f'(z)| \le \sqrt{\frac{A_0}{\pi}}\ \frac{1}{1-r^2}, \qquad |z| = r,$$

where A_0 = area of $f(U) < \infty$. The inequality (1.2.3) is sharp and the extremal function depends on r.

Iliev [54] extended Nevanlinna's result (1.2.2) by considering the functional $|(f(z_1)-f(z_2))/(z_1-z_2)|$ and proved that for $f \in S$ and points z_1, $z_2 \in U$ with $|z_1| < |z_2|$,

$$\frac{1-|z_1z_2|}{(1+|z_1|)^2(1+|z_2|)^2} \le \left| \frac{f(z_1)-f(z_2)}{z_1-z_2} \right| \le \frac{1-|z_1z_2|}{(1-|z_1|)^2(1-|z_2|)^2}$$

where the left hand side inequality holds if the segment joining $f(z_1)$ and $f(z_2)$ lies in $f(U)$ while the right hand side inequality

holds if $|z|$ either increases or decreases on the line segment joining z_1 and z_2.

Let M be a set of functions f analytic in U. The Koebe domain for the set M is denoted by $K(M)$ and is the collection of points w such that w is in f(U) for every f in M. In symbols

$$K(M) = \bigcap_{f \in M} f(U).$$

Bieberbach [11] proved that, the Koebe domain for S,

$$K(S) = \{w \in \mathbb{C} : |w| < 1/4\},$$

which settles a conjecture of Koebe [64].

The bounds on arg $f'(z)$ give information about the local rotation factor of a conformal mapping $f(z)$. Such bounds are called rotation bounds.

Goluzin [35], found that, for $f \in S$,

$$|\arg f'(z)| \le \begin{cases} 4 \operatorname{Arc} \sin r & r < 1/\sqrt{2} \\ \pi + \log (r^2/(1-r^2)) & r \ge 1/\sqrt{2} \end{cases}$$

in the disc $|z| = r < 1$.

A domain Ω in the complex plane is said to be <u>starlike</u> with respect to a point $w_o \in \Omega$, if the line segment joining w_o to every other point $w \in \Omega$ lies in Ω.

<u>Definition</u> 1.2.1 A function $f \in A_1$ is said to be starlike with respect to a point w_o if f maps U onto a domain that is starlike with respect to the point w_o.

The class of starlike functions with respect to the origin is denoted by S^*. It is known [26] that $S^* \subseteq S$.

Nevanlinna [91] found that a function $f \in A_1$ is in S^*, if and only if, for $z \in U$,

$$\text{(1.2.4)} \qquad \operatorname{Re} \frac{zf'(z)}{f(z)} > 0.$$

Geometrically, (1.2.4) gives that, for each fixed r, $0 < r < 1$, $\arg f(re^{i\theta})$ strictly increases with θ, $0 \le \theta < 2\pi$.

The growth and distortion inequalities (1.2.1) and (1.2.2) continue to hold for functions in S^* and are sharp only for the Koebe function $k(z)$ and its rotations. The Koebe domain for S^*,

$$K(S^*) = \{ w \in \mathbb{C} : |w| < 1/4 \}$$

and is thus the same as that for S. The rotation bounds for functions in S^* are due to Stroganoff [147] and Goodman [40].

Nevanlinna [91] determined that a necessary condition for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*$ is

$$\text{(1.2.5)} \qquad |a_n| \le n, \qquad n = 2, 3, \ldots$$

Strict inequality holds in (1.2.5) for all n unless f is a rotation of the Koebe function, $k(z) = z(1-z)^{-2}$. A sufficient condition for f to be in S^* is that [63],

$$\text{(1.2.6)} \qquad \sum_{n=2}^{\infty} n|a_n| \le 1.$$

The class of starlike functions is generalized by Robertson

[111], by introducing the concept of starlike functions of order α.

<u>Definition</u> 1.2.2 A function $f \in \mathcal{A}_1$ is said to be starlike of order α $(0 \le \alpha \le 1)$, if for $z \in U$,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \ge \alpha. \tag{1.2.7}$$

We denote by $S^*(\alpha)$, the class of starlike functions of order α. It follows that $S^*(0) = S^*$, $S^*(\alpha) \subseteq S^*$, $0 \le \alpha \le 1$, $S^*(1) = \{z\}$ and $S^*(\alpha_1) \subseteq S^*(\alpha_2)$ for $\alpha_2 \le \alpha_1$.

For a function $f \in S^*(\alpha)$, Robertson [111] found the bounds on growth of $|f(z)|$. Thus, for $z \in U$,

$$\frac{r}{(1+r)^{2(1-\alpha)}} \le |f(z)| \le \frac{r}{(1-r)^{2(1-\alpha)}}, \qquad |z| = r. \tag{1.2.8}$$

The function $k(z,\alpha) = z/(1-z)^{2(1-\alpha)}$ gives sharpness in (1.2.8). The sharp inequality (1.2.8) gives that the Koebe domain for $S^*(\alpha)$ is

$$K(S^*(\alpha)) = \{ w \in \mathbb{C} : |w| < 4^{\alpha-1} \}.$$

The rotation bounds for $S^*(\alpha)$ are due to Pinchuk [101].

Robertson [111] discovered that a necessary condition for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ to be in $S^*(\alpha)$ is

$$|a_n| \le \frac{1}{(n-1)!} \prod_{k=2}^{n} (k-2\alpha), \qquad n = 2, 3, \ldots \tag{1.2.9}$$

The inequality (1.2.9) is sharp for the function $k(z,\alpha)$. A sufficient condition for f to be in $S^*(\alpha)$ is [79]

$$(1.2.10) \qquad \sum_{n=2}^{\infty} (n-\alpha) \left|a_n\right| \leq 1 - \alpha.$$

A domain Ω in the complex plane is said to be <u>convex</u> if it contains the line segment joining z_1 and z_2 for all distinct points z_1, z_2 in Ω.

<u>Definition</u> 1.2.3 A function $f \in A_1$ is said to be convex if f maps U onto a convex domain.

We denote the class of convex functions by CV. Study [149] and Robertson [111] found an analytic characterization of functions in CV. Thus, a function $f \in A_1$ is in CV, if and only if, for $z \in U$,

$$(1.2.11) \qquad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0.$$

Geometrically, the condition (1.2.11) means that $w = f(re^{i\theta})$ maps each circle $|z| = r < 1$ onto a simple closed contour whose tangent rotates monotonically as θ increases in the counter clockwise direction.

It is observed that $CV \subseteq S^*$. The containment is proper since the Koebe function $k(z) = z/(1-z)^2$ is in S^*, but is not in the class CV.

Alexander [4] discovered a close connection between the classes CV and S^*. Thus, a function

$$(1.2.12) \qquad f \in CV, \text{ if and only if, } zf' \in S^*.$$

The kind of relation (1.2.12) is called Alexander type relation.

The growth bounds for $f \in CV$ are determined by Grownwall [48]. Thus, for $z \in U$,

(1.2.13) $$\frac{r}{1+r} \le |f(z)| \le \frac{r}{1-r}, \qquad |z| = r.$$

He also determined distortion bounds for CV. Thus, for $z \in U$,

(1.2.14) $$\frac{1}{(1+r)^2} \le |f'(z)| \le \frac{1}{(1-r)^2}, \qquad |z| = r.$$

The function $l(z) = z/(1-z)$ and its rotations only give sharpness in the inequalities (1.2.13) and (1.2.14). Loewner [70] independently obtained (1.2.13) and (1.2.14).

For the class CV, the bounds on the functional $|(f(z_1)-f(z_2))/(z_1-z_2)|$ are determined by Iliev [54]. Thus for $f \in CV$ and points z_1, $z_2 \in U$ with $|z_1| < |z_2|$,

(1.2.15) $$\frac{1}{(1+|z_1|)(1+|z_2|)} \le \left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \le \frac{1}{(1-|z_1|)(1-|z_2|)}$$

where the right hand side inequality holds if $|z|$ either increases or decreases on the line segment joining z_1 and z_2.

The Koebe domain for CV,

$$K(CV) = \{ w \in \mathbb{C} : |w| < 1/2 \}.$$

It is known [26] that the rotation bounds for CV are given by,

$$|\arg f'(z)| \le 2 \text{ Arc} \sin r, \qquad |z| = r,$$

for $z \in U$ and the inequality is sharp.

A necessary condition for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV$ is

(1.2.16) $$|a_n| \le 1, \qquad n = 2,3, \ldots .$$

obtained by Loewner [70]. Strict inequality holds for all n in (1.2.16) unless f is a rotation of the function $l(z) = z/(1-z)$. It is known [63] that a sufficient condition for f to be in CV is

(1.2.17) $$\sum_{n=2}^{\infty} n^2 |a_n| \le 1.$$

The class of convex functions is also generalized by Robertson [111] by introducing the concept of convex functions of order α.

<u>Definition</u> 1.2.4 A function $f \in A_1$ is said to be convex of order α $(0 \le \alpha \le 1)$, if and only if, for $z \in U$,

(1.2.18) $$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \ge \alpha .$$

We denote the class of convex functions of order α by $CV(\alpha)$. Brown [21] has found an analytic condition that is equivalent to (1.2.18). Thus, a function f belonging to S, is in $CV(\alpha)$ $(0 \le \alpha < 1)$, if and only if, $\mathrm{Re}\,(1+(z-z_o)f''(z)/f'(z)) > \alpha$ for $|z - z_o| < 1 - |z_o|$, $|z_o| < 1$. It is easily seen that $CV(0) = CV$, $CV(\alpha) \subseteq CV$, $0 \le \alpha \le 1$, $CV\{1\} = \{z\}$ and $CV(\alpha_1) \subseteq CV(\alpha_2)$ for $\alpha_2 \le \alpha_1$. An Alexander type relation connects $CV(\alpha)$ and $S^*(\alpha)$. Thus, for $0 \le \alpha \le 1$,

$$f \in CV(\alpha), \text{ if and only if, } zf' \in S^*(\alpha).$$

Allowing α to be negative also in Definition 1.2.4, Hallenbeck [51] showed that $CV(\alpha)$ also consists of univalent functions for α satisfying $-1/2 < \alpha < 0$.

Strohhacker [148] and Marx [76] proved that every convex function is starlike of order 1/2 and that this result is sharp. Later, Jack [55] proposed a general problem, viz. if $f \in CV(\alpha)$, find $\beta(\alpha)$ such that $f \in S^*(\beta(\alpha))$. He obtained a partial solution to this problem. MacGregor [74] and Goel [33] solved this problem completely by showing that if f is in $CV(\alpha)$ $(0 \leq \alpha < 1)$ then f is in $S^*(\beta)$ where

$$\beta = \frac{1-2\alpha}{2^{2(1-\alpha)}}(1-2^{2\alpha-1})^{-1} \qquad \text{if } \alpha \neq 1/2,$$

$$\beta = \frac{1}{2\log 2} \qquad \text{if } \alpha = 1/2.$$

The growth bounds of functions $f \in CV(\alpha)$, $0 \leq \alpha \leq 1$, are obtained by Robertson [111]. Thus, for $z \in U$, $|z| = r$,

$$(1.2.19) \quad \begin{cases} \dfrac{(1+r)^{2\alpha-1}-1}{2\alpha-1} \leq |f(z)| \leq \dfrac{1-(1-r)^{2\alpha-1}}{2\alpha-1}, & \text{if } \alpha \neq 1/2 \\ \ln(1+r) \leq |f(z)| \leq -\ln(1-r), & \text{if } \alpha = 1/2. \end{cases}$$

The function

$$(1.2.20) \quad l_\alpha(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \text{if } \alpha \neq 1/2 \\ -\log(1-z), & \text{if } \alpha = 1/2, \end{cases}$$

gives sharpness in (1.2.19). Robertson [111] also found distortion bounds for $CV(\alpha)$. Thus, for $z \in U$,

$$(1.2.21) \qquad \frac{1}{(1+r)^{2(1-\alpha)}} \leq |f'(z)| \leq \frac{1}{(1-r)^{2(1-\alpha)}}.$$

The function $l_\alpha(z)$ in (1.2.20) gives sharpness in (1.2.21). It follows from (1.2.19) that the Koebe domain for $CV(\alpha)$,

$$K\ (CV(\alpha)) = \begin{cases} \{ w: |w| < (2^{2\alpha-1} - 1)/(2\alpha - 1)\}, & \text{if } \alpha \neq 1/2, \\ \{ w: |w| < \ln 2 \}, & \text{if } \alpha = 1/2. \end{cases}$$

The rotation bounds for $CV(\alpha)$ are due to Pinchuk [101]. Thus, for $z \in U$, and $f \in CV(\alpha)$, $0 \le \alpha \le 1$,

$$(1.2.22) \qquad |\arg f'(z)| \le 2(1-\alpha) \operatorname{Arc} \sin r, \qquad |z| = r.$$

The inequality (1.2.22) is sharp.

Robertson [111] determined that for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $CV(\alpha)$, $0 \le \alpha \le 1$,

$$(1.2.23) \qquad |a_n| \le \frac{1}{n!} \prod_{k=2}^{n} (k-2\alpha).$$

The inequality (1.2.23) is sharp for the function $l_\alpha(z)$ in (1.2.20). It follows from (1.2.10) that a sufficient condition for f to be in $CV(\alpha)$ is

$$(1.2.24) \qquad \sum_{n=2}^{\infty} n(n-\alpha)\, |a_n| \le 1-\alpha.$$

For an analytic and locally univalent function $f(z)$ in U, and $r \in (0,1)$, the (Euclidean) curvature, $k(f;z)$ of the curve $f(|z| = r)$ at the point $f(z)$, is given by [149]

$$(1.2.25) \qquad k(f;z) = \frac{\operatorname{Re}\,(1+zf''(z)/f'(z))}{|zf'(z)|}, \qquad |z| = r.$$

The radius of curvature $\rho(f;z)$ of the curve $f(|z|=r)$ at the point $f(z)$ is defined as,

$$(1.2.26) \qquad \rho(f;z) = \frac{1}{k(f;z)} .$$

We denote, throughout in the sequel,

$$(1.2.27) \qquad k(z) = k(f;z), \qquad \rho(z) = \rho(f;z).$$

In geometrical terms, $k(f;z) = ds/d\psi$ where s is the arc length on $f(|z| = r)$ and ψ is the angle the tangent to $f(|z| = r)$ makes with the positive real axis.

If $f \in S$, it is known that [42], for $z \in U$,

$$(1.2.28) \qquad \frac{1-4r+r^2}{1-r^2} \le \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \le \frac{1+4r+r^2}{1-r^2}, \qquad |z| = r.$$

Using (1.2.2) and (1.2.28), it follows that for $z \in U$,

$$(1.2.29) \qquad \left(\frac{1-r}{1+r}\right)^2 \frac{1-4r+r^2}{r} \le k(f;z) \le \left(\frac{1+r}{1-r}\right)^2 \frac{1+4r+r^2}{r}, \quad |z| = r.$$

The lower bound in (1.2.29) is sharp and changes sign at the radius of convexity $r_{cv} = 2 - \sqrt{3}$, [65], [83].

Goodman found [41] that if $f \in S^*$, $z \in U$ and Γ is a directed curve through z, then

$$(1.2.30) \qquad |k(f;z)| \le \frac{(1+r)^3}{1-r} |K_u(\Gamma)| + (4-2r)\left(\frac{1+r}{1-r}\right)^2, \qquad |z| = r$$

where $K_u(\Gamma) = K_u(\Gamma(z))$ is the curvature of Γ at z. Equality occurs in (1.2.30) for a suitable rotation of the Koebe function.

It is clear that for $f \in CV$, $k(f;z) > 0$ in $U\backslash\{0\}$. Zmorovic [162] proved that if $f \in CV$, then the bounds on $k(f;z)$ in terms

of $|z|$ are given by

$$(1.2.31) \qquad \frac{1-r^2}{r} \le k(f;z) \le \frac{1-r^2}{r} \frac{2}{t} \sin \frac{t}{2} \exp\left(-1 + \frac{t}{2} + \frac{t}{e^t - 1}\right)$$

where $t = 2 \ln \frac{1+r}{1-r}$ and $|z| = r < 1$. The inequality (1.2.31) is sharp for the function $f(z) = (1 - (\frac{1-z}{1+z})^{1-2\lambda}) \frac{1}{2-4\lambda}$, where $0 \le \lambda = t^{-1} - (e^t - 1)^{-1} \le 1/2$.

Eenigenburg [27] found the lower bound on $k(f;z)$ if f is in $CV(\alpha)$, $0 \le \alpha < 1$. Thus,

$$k(f;z) \ge \frac{(1-r^2)^{1-\alpha}}{r}, \qquad |z| = r < 1,$$

with equality holding only for the function (and rotations) $l_\alpha(z)$ given in (1.2.20).

Korickii [65] found that the curvature $K_{\phi'}$ of the image curve of the radial line $\arg z = \phi'$ under functions in CV is bounded. In [82] Mirosnicenko describes the extremal functions for which $K_{\phi'}$ attains maximum for functions in the class S. A sharp lower bound on curvature K_r of $f(|z| = r)$ is also found in [82] when f is an odd univalent function in U.

Recently Goodman [44] introduced a very interesting concept of convex functions of bounded type based on curvature of $f(|z| = r)$, $0 < r < 1$. Roughly, these are the normalized univalent functions with the radius of curvature $\rho(f;z)$ bounded as $R_1 \le \rho(f;z) \le R_2$, $0 \le R_1 \le R_2 \le \infty$ on $|z| = 1$. To give a precise definition of convex functions of bounded type, let

$$(1.2.32) \qquad \rho_*(r) = \min_{|z|=r} \rho(z), \qquad \rho^*(r) = \max_{|z|=r} \rho(z)$$

and

$$(1.2.33) \qquad R_* = \liminf_{r\to 1^-} \rho_*(r), \qquad R^* = \limsup_{r\to 1^-} \rho^*(r).$$

Limit superior in the definition of R^* may be replaced by limit since $\rho^*(r)/r$ is increasing [157] in r on (0,1).

<u>Definition</u> 1.2.5 Let R_1, R_2 be fixed in $[0, \infty]$. A function $f \in S$ is said to be in the class $CV(R_1,R_2)$ if $R_1 \le R_*$ and $R^* \le R_2$ where R_* and R^* are as in (1.2.33). For $0 < R_1 \le R_2 < \infty$, a function f in $CV(R_1,R_2)$ is called a <u>convex function of bounded type</u>.

A function f is said to be in $\overline{CV}(R_1,R_2)$ if, $R_1 = R_*$, $R_2 = R^*$ where R_* and R^* are as in (1.2.33).

For $0 \le R_1^* \le R_1 \le R_2 \le R_2^* \le \infty, R_2 \ge 1$, $CV(R_1,R_2) \subseteq CV(R_1^*,R_2^*)$. It follows [44] that $\bigcup_{0\le R_1\le R_2\le\infty} CV(R_1,R_2) = CV$. In [44], Goodman found that for $R_2 < \infty$, $CV(R_1,R_2) \subseteq CV(\alpha)$ for some $\alpha > 1/4R_2$ and that $CV(\beta)$ is not contained in $CV(R_1,R_2)$ for $0 \le \beta < 1$.

Usually, for the problems concerning $CV(R_1,R_2)$ the function

$$(1.2.34) \qquad F_{R_2}(z) = \begin{cases} \dfrac{z}{1-\sqrt{1-1/R_2}\, z}, & 1 \le R_2 < \infty \\[2ex] \dfrac{z}{1-z}, & R_2 = \infty \end{cases}$$

behaves as an extremal function. For $1 \le R_2 < \infty$, F_{R_2} maps U

conformally onto the disc having centre at $\sqrt{R_2^2 - R_2}$ and radius R_2. F_∞ maps U conformally onto $\{ w \in \mathbb{C}: \operatorname{Re} w > -1/2 \}$. Clearly, for $1 \le R_2 \le \infty$, F_{R_2} is in $CV(R_1,R_2)$, for any R_1 such that $0 \le R_1 \le R_2$. In the case $R_2 = \infty$, quantities like $R_2 - \sqrt{R_2^2 - R_2}$ have to be interpreted as their limits as R_2 tends to ∞.

Some other examples of functions in $CV(R_1,R_2)$ for specific choices of R_1 and R_2 are as follows:

(1.2.35) $$f_1(z) = \left(1 - \left(\frac{1-z}{1+z}\right)^{1-2\lambda}\right)\frac{1}{2-4\lambda} \in \overline{CV}(\infty,\infty),\ 0\le\lambda<1/2.$$

$$f_2(z) = \begin{cases} \frac{1}{a}\log\frac{1}{1-az} \in \overline{CV}(1,(1-a^2)^{-1/2}), & 0<a<1. \\ \log\frac{1}{1-z} \in \overline{CV}(1,\infty) \end{cases}$$

$$f_3(z) = e^z - 1 \in \overline{CV}(1,\infty)$$

$$f_4(z) = \frac{1}{2}\log\frac{1+z}{1-z} \in \overline{CV}(1,\infty)$$

$$h_2(z) = z+z^2/4 \in \overline{CV}(\sqrt{3}/2,\infty)$$

$$h_3(z) = z+z^3/9 \in \overline{CV}(\sqrt{3}/2,\infty)$$

$$h_{a,3}(z) = z+az^3 \in \overline{CV}\left(\frac{(1+3a)^2}{1+9a}, \frac{(1-3a)^2}{1-9a}\right),\ 0\le a<1/15$$

$$h_k(z) = z+z^k/k^2 \in \overline{CV}\left(\frac{(k+1)^2}{2k^2}, \infty\right),\ k \ge 4$$

$$h_{a,k}(z) = z+az^k \in \overline{CV}(R_1(k,a),\ R_2(k,a)),\ 0<a<1/k^2 \text{ for } k=2;$$

$$1/15 \le a < 1/9 \text{ for } k = 3 \text{ and}$$

$$R_1(k,a) = \left(\frac{3}{k+1}\right)^{3/2}(k-1)^{1/2}(1-k^2a^2)^{1/2},\ R_2(k,a) = \frac{(1-ka)^2}{1-k^2a}.$$

$$h_{a,k}(z) = z+az^k \in \overline{CV}\left(\frac{(1+ka)^2}{1+k^2a}, \frac{(1-ka)^2}{1-k^2a}\right), 0 \le a < 1/k^2, k \ge 4.$$

For a function $f \in CV(R_1,R_2)$, it follows [45] that

(1.2.36) $$\text{Area } (f(U)) \le \pi R_2^{\,2}.$$

Using (1.2.36), Goodman [44] determined that for

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV(R_1,R_2)$,

(1.2.37) $$1 + \sum_{n=2}^{\infty} n|a_n|^2 \le R_2^{\,2}$$

and equality holds for F_{R_2} given by (1.2.34), $R_2 \ge 1$. It follows from (1.2.37) that for $k \ge 2$,

(1.2.38) $$|a_k| \le \left(\frac{R_2^{\,2}-1}{k} \right)^{1/2}.$$

It is not known whether the inequality (1.2.38) is actually sharp. The function F_{R_2} given by (1.2.34) suggests that for $R_2 \ge 1$, $k \ge 2$

(1.2.39) $$|a_k| \le (1-1/R_2)^{(k-1)/2}$$

may be true. Raupach [106] proved (1.2.39) for $k = 2$. An independent proof of (1.2.39) for $k = 2$, was given by Wirths [159]. Mejia and Minda [78] also established (1.2.39) for $k = 2$. Wirths [159] proved (1.2.39) for $k = 3$. The binomials $p_{a,n}(z) = z + az^n$, $a \ge 0$, $n \ge 2$, $0 \le an^2 < 1$ given by (1.2.35) show that (1.2.39) is not true for <u>all</u> $k \ge 2$ and $R_2 > 1$.

Goodman [44], [46] obtained upper growth bounds for f in

$CV(R_1,R_2)$ in terms of distance of boundary of $f(U)$ from the origin. Thus, for $z \in \bar{U}$,

$$|f(z)| \le 2R_2 - d \tag{1.2.40}$$

and

$$|f(z)| \le \frac{rd\,(2R_2-d)}{R_2(1-r)+rd} \tag{1.2.41}$$

where $d = \inf\limits_{\zeta\in\partial f(U)} |\zeta|$. Both the inequalities (1.2.40) and (1.2.41) are sharp for F_{R_2} given by (1.2.34). For $R_2 < \infty$, $\overline{f(U)}$ is homeomorphic image of $\bar{U}$ under $f(z)$. For $R_2 = \infty$, the upper bounds in (1.2.40), (1.2.41) have to be interpreted as limits as R_2 tends to ∞ and if $f(e^{i\theta})$ for some real θ does not exist, we assume that $|f(e^{i\theta})| = \infty$.

Ma et al. [71] found both lower and upper growth bounds independent of d for $f \in CV(R_1,R_2)$, $R_2 < \infty$. Thus, for $z \in U$,

$$\frac{r}{1+r\sqrt{1-1/R_2}} \le |f(z)| \le \frac{r}{1-r\sqrt{1-1/R_2}}, \qquad |z| = r. \tag{1.2.42}$$

The sharp upper bound in the inequality (1.2.42) was first obtained by Goodman [46] in the annulus $1>|z| \ge 2\sqrt{R_2{}^2-R_2}/(2R_2-1)$.

For $f \in CV(R_1,R_2)$ an upper distortion bound is due to Goodman [46] for which the sharp function depends on z. Thus,

$$|f'(z)| \le \frac{R_2}{1-r^2}, \qquad |z| = r < 1. \tag{1.2.43}$$

The inequality (1.2.43) is sharp for each $r \in (0,1)$ and F_{R_2} given

by (1.2.34), where $R_2 = 1/(1-r^2)$.

It follows that [71] the sharp lower and upper distortion bounds for $f \in CV(R_1,R_2)$, $R_2 < \infty$, are as follows:

$$\frac{1}{(1+r\sqrt{1-1/R_2})^2} \le |f'(z)| \le \frac{1}{(1-r\sqrt{1-1/R_2})^2}, \qquad |z| = r \tag{1.2.44}$$

where the lower bound holds for $z \in U$ and the upper bound holds for z in disc $|z| \le 2/(1+\sqrt{5-4A})$, $A = \sqrt{1-1/R_2}$. Equality holds in any one of the inequalities in (1.2.42) and (1.2.44) for $z \neq 0$, if and only if, f is a rotation of the function F_{R_2} in (1.2.34). Unlike in the sharpness function for (1.2.43), in the sharpness function F_{R_2} for (1.2.44), R_2 is independent of r.

Further for $f \in CV(R_1,R_2)$ and z in the annulus $1 > |z| > \mathfrak{r}$, $\mathfrak{r} = 2/(1+\sqrt{5-4A})$, $A = \sqrt{1-1/R_2}$, an upper distortion bound better than that in (1.2.43) is due to Ma et al. [71]. Thus

$$|f'(z)| \le \frac{R_2(1+\sqrt{1-1/R_2})}{(1-r^2)r\ (2+(1-\sqrt{1-1/R_2})\ r)}, \qquad |z| = r. \tag{1.2.45}$$

For $f \in CV(R_1,R_2)$ an upper distortion bound involving $\delta(f(z))$ = the distance of $\partial f(U)$ from the point $f(z)$, is found by Mejia and Minda [78]. Thus, for $z \in U$,

$$|f'(z)| \le \frac{\delta(f(z))\ [2R_2-\delta(f(z))]}{R_2\ (1-r^2)}, \qquad |z| = r. \tag{1.2.46}$$

Equality holds in (1.2.46), only for conformal functions that map U onto a disc of radius R_2.

Goodman [44] obtained bounds on the distance of $\partial f(U)$ from the origin when $f \in CV(R_1,R_2)$. Thus,

$$R_2 - \sqrt{R_2^2 - R_2} \le d \le R_1 - \sqrt{R_1^2 - R_1} \tag{1.2.47}$$

where the upper bound holds when $R_1 \ge 1$ and $d = \inf_{\zeta \in \partial f(U)} |\zeta|$. Further, it follows [44] that bounds on R_1 and R_2 in terms of d are

$$R_1 \le \frac{d^2}{2d-1} \le R_2 \tag{1.2.48}$$

where the lower inequality holds when $R_1 \ge 1$ and that the Koebe domain for $CV(R_1,R_2)$ is given by

$$K(CV(R_1,R_2)) = \{ w \in \mathbb{C} : |w| < R_2 - \sqrt{R_2^2 - R_2} \}.$$

For $f \in CV(R_1,R_2)$, the inequalities (1.2.40), (1.2.47) give the upper bound on the distance of farthermost point on $\partial f(U)$ from the origin. Thus,

$$d^* \le R_2 + \sqrt{R_2^2 - R_2} \, , \tag{1.2.49}$$

where $d^* = \sup_{\zeta \in \partial f(U)} |\zeta|$. The function F_{R_2} in (1.2.34) gives sharpness in the inequalities (1.2.47) through (1.2.49).

Goodman [44] introduced the concept of starlike functions of bounded type. He defined a class $ST(R_1,R_2)$, $0 \le R_1 \le R_2 \le \infty$, $R_2 \ge 1$, by connecting it to the class $CV(R_1,R_2)$ through an Alexander type relation. Thus $f \in CV(R_1,R_2)$, if and only if, zf' is in $ST(R_1,R_2)$. For $0 < R_1 \le R_2 < \infty$, functions in $ST(R_1,R_2)$ are

called starlike functions of bounded type.

In 1956, the following fundamental theorem was found:

<u>Blaschke's Rolling Theorem</u> [67] Given a regular, simple, complete convex curve C, the circle of radius $(\sup_{z\in C} k(z))^{-1}$ rolls freely inside C in the sense that, if it touches C from inside at any point, it lies entirely within the closed convex set bounded by C and it is the largest circle with this property, where $k(z)$ is curvature. The smallest circle with the property that C with $\inf_{z\in C} k(z) > 0$ rolls freely inside it, has radius $(\inf_{z\in C} k(z))^{-1}$.

Inspired by Blaschke's Rolling Theorem, Styer and Wright [150] attempted to generalize the class $CV(R_1,R_2)$ by introducing the class $CVG(R_1,R_2)$, $0 \le R_1 \le R_2 \le \infty$, $R_2 \ge 1$. Roughly, a normalized univalent function $f \in CVG(R_1,R_2)$, if and only if, a circle of radius R_1 can roll around the inside of $\partial f(U)$ and a circle of radius R_2 containing $f(U)$ in its interior can roll around the outside of $\partial f(U)$.

The precise definition of the class $CVG(R_1,R_2)$ is given as follows:

<u>Definition</u> 1.2.6 Given $0 \le R_1 \le R_2 \le \infty$, $R_2 \ge 1$, let $CVG(R_1,R_2)$ be the class of functions f in S with the property that for each $\eta \in \partial f(U)$ there are open discs $D_1(\eta)$ and $D_2(\eta)$ of radius R_1 and R_2 respectively such that, $\eta \in \partial D_1(\eta) \cap \partial D_2(\eta)$ and

$$D_1(\eta) \subseteq f(U) \subseteq D_2(\eta).$$

If $R_1 = 0$ or $R_2 = \infty$, $D_1(\eta)$ and $D_2(\eta)$ are to be interpreted as the empty set and an open half-plane, respectively and in the case $R_1 = 0$, the condition $\eta \in \partial D_1(\eta) \cap \partial D_2(\eta)$ is to be replaced by $\eta \in \partial D_2(\eta)$.

It follows [78], [150] that $CVG(0,R_2) = CV(0,R_2)$ for $R_2 \geq 1$. Styer and Wright [150] obtained that $CV(R_1,R_2) \subseteq CVG(R_1,R_2) \subseteq CV$ for $0 \leq R_1 \leq R_2 \leq \infty$, $R_2 \geq 1$. For $0 \leq R_1^* \leq R_1 \leq R_2 \leq R_2^* \leq \infty$, it is easily seen that $CVG(R_1,R_2) \subseteq CVG(R_1^*,R_2^*)$. Thus it follows for $0 \leq R_1 \leq R_2$, $R_2 \geq 1$, that

$$CVG(R_1,R_2) \subseteq CV(0,R_2). \tag{1.2.50}$$

It is observed [150] that the inequalities (1.2.47), (1.2.49) continue to hold for $CVG(R_1,R_2)$ and equality holds in these inequalities for a function f, if and only if, f is a rotation of F_{R_2} given by (1.2.34). Thus, the Koebe domain for $CVG(R_1,R_2)$,

$$K(CVG(R_1,R_2)) = \{ w \in \mathbb{C}: |w| < R_2 - \sqrt{R_2^2-R_2} \}$$

is the same as that for $CV(R_1,R_2)$. It follows from (1.2.50) and [159] that the inequality (1.2.39) continues to hold for $CVG(R_1,R_2)$, for $k = 2$ and 3.

For a function f in S, the unit exterior normal to the curve $f(|z| = r)$, $r \in (0,1)$ at the point $f(z)$ is given by

$$n(z) = \frac{zf'(z)}{|zf'(z)|}. \tag{1.2.51}$$

Styer and Wright [150] found that for $0 \leq R_1 \leq R_2 \leq \infty$, $R_2 \geq 1$ and $f \in S$, the following are equivalent.

(i) $f \in CVG(R_1, R_2)$.

(ii) $f \in CV$, and for every $\zeta \in \partial U$ for which $f(\zeta)$ is finite,

(1.2.52) $$f(U) \subseteq D(f(\zeta) - R_2 n(\zeta),\ R_2).$$

Further, in the case $R_1 > 0$,

(1.2.53) $$D(f(\zeta) - R_1 n(\zeta),\ R_1) \subseteq f(U)$$

where $n(\zeta)$ is as in (1.2.51) and $D(a,R) = \{ z \in \mathbb{C}: |z-a| < R \}$.

(iii) $f(U)$ is the intersection of open discs of radius R_2 and in the case $R_1 > 0$, the union of open discs of radius R_1.

Let $a,b \in \mathbb{C}$, $|a-b| \le 2R$, $R > 0$. For $|a-b| < 2R$, let $E(a,b;R)$ denote $\Delta_1 \cap \Delta_2$ where Δ_1 and Δ_2 are open discs of radius R such that $a,\ b \in \partial\Delta_1 \cap \partial\Delta_2$ and for $|a-b| = 2R$, set

$$E(a,b;R) = D((a+b)/2,\ R).$$

It is proved in [150] that for $1 \le R_2 \le \infty$, $R_1 = 0$ and $f \in S$, the above conditions (i), (ii) and (iii) are equivalent to

(iv) for every $u,v \in f(U)$, $\exists\ a,b \in f(U)$ such that

$$u,v \in E(a,b;R_2) \subseteq f(U).$$

A class of locally univalent functions that contains the class $CV(R_1,R_2)$ was studied by Wirths [159],[160].

<u>Definition</u> 1.2.7 A function $f \in \mathcal{A}$ is said to be in the class $C(K)$, $K > 0$, if and only if, f is locally univalent in U and

$$\liminf_{|z|\to 1} k(f;z) \ge K$$

where $k(f;z)$ is as in (1.2.25).

Clearly, for $R_2 < \infty$, $CV(R_1,R_2) \subseteq C(1/R_2)$ and for $0 < K \le K^*$, the class $C(K^*) \subseteq C(K)$. Functions $f \in C(K)$, satisfy the following local minimum property [159]:

(1.2.54) $k(f;z)$ has no local minimum in $U\setminus\{0\}$, $k(f;z) > K$ in $U\setminus\{0\}$.

The analogues of the inequality (1.2.39) for functions in the class $C(K)$ have been proved for $k = 2$, in [159], [160] and for $k = 3$ in [159].

Further, the Taylor coefficients c_k, for $k \ge 1$ of the function $f(z) = \sum_{k=0}^{\infty} c_k z^k \in C(K)$, $K > 0$, satisfy the following relation [160],

(1.2.55) $$K\left|c_1 - \frac{1}{2}(n-1)\, c_{n-1}a^2 + nc_n a - \frac{1}{2}(n+1)\, c_{n+1}\right|$$
$$\le 1+a^2 - a \operatorname{Re} E_1 - \frac{1}{2}(1+a^2) \operatorname{Re} E_n + \frac{1}{2} a \operatorname{Re} (E_{n-1}+E_{n+1}),$$

where $n \ge 2$, $1+zf''(z)/f'(z) = \sum_{k=0}^{\infty} E_k z^k$ and $a = \sqrt{1-K|c_1|}$. The inequality (1.2.55) is sharp.

Another interesting class which is closely related to the class $CV(R_1,R_2)$ is recently introduced by Mejia and Minda [78] as follows.

Let $k \in [0,\infty)$. A region $\Omega \subseteq \mathbb{C}$ is called k-convex if $|a-b| < 2/k$ for any pair of distinct points a, $b \in \Omega$ and the intersection $E(a,b;1/k)$ of the two open discs of radii $1/k$ that both have a and b on their boundary lies in Ω. When $k = 0$,

$E(a,b;\infty)$ is simply $[a,b]$, the straight line segment between the points a and b.

The following characterization for functions $f \in CV(0,R_2)$, in terms of the $1/R_2$ - convex regions is obtained in [78]:

$$(1.2.56)\quad \left\{ \begin{array}{l} \text{If } 1 \le R_2 \le \infty \text{ and } f \in S, \text{ then } f \in CV(0,R_2), \text{ if} \\ \text{and only if, } f(U) \text{ is a } 1/R_2\text{-convex region.} \end{array} \right.$$

<u>Definition</u> 1.2.8 Let for $k \ge 0$, $K(k,\alpha)$ denote the class of all analytic functions f on U such that f is univalent in $U, f(0) = 0$, $f'(0) = \alpha > 0$ and $f(U)$ is a k-convex region.

The normalization in the definition forces $\alpha \le 1/k$ [78]. It is observed that [78] $K(0,1) = CV$ and for $k \le 1, CV(0,1/k) \subseteq K(k,1)$.

Ma et al. [71] obtained growth and distortion bounds for $K(k,\alpha)$, $k > 0$, analogous to that in the inequalities (1.2.42), (1.2.44), (1.2.45). Distortion bounds similar to that in the inequality (1.2.46) and Koebe domain for $K(k,\alpha)$, $k \ge 0$ were obtained by Mejia and Minda [78]. The sharp bound on $|f''(0)|/2!$ and the following basic estimate for $f \in K(k,\alpha)$, $k \ge 0$, were also found in [78]: For $z \in U$,

$$(1.2.57)\qquad \left| \frac{f''(z)(1-|z|^2)}{2f'(z)} - \bar{z} \right| \le \sqrt{1-(1-|z|^2)\,|f'(z)|k}\,.$$

Equality holds in (1.2.57) at a point, if and only if, f is a rotation of the function $\alpha z/(1-\sqrt{1-\alpha k}\, z)$.

1.3 A function $f \in \mathcal{A}$ is said to be subordinate in U to a function $F \in \mathcal{A}$ (denoted by $f(z) \ll F(z)$) if there exists a function $w \in \mathcal{A}$ such that $|w(z)| \leq |z|$ and $f(z) = F(w(z))$ in U.

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $F(z) = \sum_{n=1}^{\infty} b_n z^n$ be in $\mathcal{A}$ and $f(z) \ll F(z)$. Clearly it follows from the definition of subordination that $|a_1| \leq |b_1|$. Further [113], [114]

$$(1.3.1) \qquad \sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |b_k|^2 , \qquad n = 1, 2, \ldots$$

We say that a function $F \in \mathcal{A}$ majorizes a function $f \in \mathcal{A}$ in a disc $|z| < R$, $0 < R \leq 1$ if $|f(z)| \leq |F(z)|$ in the disc $|z| < R$.

Biernacki [14], [15] was first to investigate connection between subordination and majorization. It is discovered [15] that if $F \in CV$ and f is convex and univalent in U, then

$$(1.3.2) \quad \begin{cases} f(z) \ll F(z) \text{ implies } |f(z)| \leq |F(z)| \text{ in the disc } |z| < R_0 \\ \text{where } R_0 \cong 0.543 \text{ is the least positive root of} \\ \operatorname{Arc}\sin x + 2 \operatorname{Arc}\tan x = \pi/2. \end{cases}$$

The connection between subordination and majorization has been further investigated in [37], [127] and [13].

The study of relation between subordination and majorization of derivatives was initiated by Goluzin [36], [37], [38]. It is known due to Shah-Tao [128] that if $F \in S$ and $f'(0) \geq 0$, then

$$(1.3.3) \qquad f(z) \ll F(z) \text{ implies } |f'(z)| \leq |F'(z)| \text{ in } |z| \leq 3-\sqrt{8}.$$

A class closely related to certain subclasses of S is P defined as follows:

<u>Definition</u> 1.3.1 A function $p \in A$ with $p(0) = 1$ is said to be in the class P if $p(z) \ll \frac{1+z}{1-z}$.

It follows from (1.2.4) that a function $f \in A_1$ is in S^*, if and only if, zf'/f is in P. Similarly, (1.2.11) gives that a function $f \in A_1$ is in CV, if and only if, $1+zf''/f'$ is in P.

Janowski [57] introduced the following generalization of the class P.

<u>Definition</u> 1.3.2 A function $p \in A$ with $p(0) = 1$ is said to be in the class $P(A,B)$, $-1 \leq B < A \leq 1$, if $p(z) \ll \frac{1+Az}{1+Bz}$.

Geometrically a function p is in $P(A,B)$, if and only if, $p(0) = 1$ and $p(U)$ is inside an open disc centred on the real axis with a diameter having end points

$$D_1 = (1-A)/(1-B), \qquad D_2 = (1+A)/(1+B),$$

if $B \neq -1$ and $p(U)$ is inside an open disc that lies in the half plane $\mathrm{Re}\, w > (1-A)/2$ if $B = -1$.

Clearly, $P(A,B) \subseteq P$. Special choices of parameters A and B lead to familiar classes:

$P(1,-1) = P$.

The class $P(1,-1 + 1/M)$, $M > 1/2$ was studied by Janowski [56].

The class $P(1,2\alpha-1)$, $0 \leq \alpha < 1$ was studied by Shaffer [125].

The class $P(1-2\alpha,-1)$, $0 \le \alpha < 1$ is the class defined by $\operatorname{Re} p(z) > \alpha$, $p(0) = 1$ and was studied by McCarty [77].

It is known [5] that if $p \in P(A,B)$, $-1 \le B < A \le 1$, then on $|z| = r < 1$,

$$\frac{1-Ar}{1-Br} \le \operatorname{Re}(p(z)) \le |p(z)| \le \frac{1+Ar}{1+Br}$$

and is sharp for $(1+Az)/(1+Bz)$.

A class $\mathcal{B}(\alpha)$ [22] closely related to the class $P(1-2\alpha,-1)$, $0 \le \alpha < 1$ is defined as follows:

Definition 1.3.3 A function $f \in A_1$ is said to be in the class $\mathcal{B}(\alpha)$, $0 \le \alpha < 1$, if and only if, the function f/z is in the class $P(1-2\alpha,-1)$.

Clearly $\mathcal{B}(\alpha) \subseteq \mathcal{B}(0)$ for $0 < \alpha < 1$. It is known that [161] if $f \in \mathcal{B}(0)$, then $f(z)$ is univalent in the disc $|z| < \sqrt{2} - 1$.

Many of the important subclasses of univalent functions may be studied conveniently in terms of the class $P(A,B)$. Janowski [57] generalized the classes $S^*(\alpha)$, $0 \le \alpha \le 1$, by introducing the following class of functions.

Definition 1.3.4 A function $f \in A_1$ is said to be in the class $S^*(A,B)$, $-1 \le B < A \le 1$, if and only if, zf'/f is in $P(A,B)$.

It is easily seen that $S^*(1,-1) = S^*$, $S^*(1-2\alpha, -1) = S^*(\alpha)$, $0 \le \alpha < 1$ and $S^*(A,B) \subseteq S^*$, $-1 \le B < A \le 1$. For $f \in S^*(A,B)$ and $|z| = r < 1$ the growth bounds are given by [5]:

$$r(1-Br)^{(A-B)/B} \le |f(z)| \le r(1+Br)^{(A-B)/B}, \quad \text{if } B \ne 0,$$

$$r \exp(-Ar) \le |f(z)| \le r \exp(Ar), \quad \text{if } B = 0.$$

The class $S^*(A,B)$ or its particular cases are studied in [141], [95], [56], [28], [57], [132], [34]. In [97] a particular integral operator on the class $S^*(A,B)$ is studied.

A "continuous" passage from starlike functions to convex functions is the following class [87].

Definition 1.3.5 A function f in $\mathcal{A}_1$ is said to be α-convex in U, $\alpha \in \mathbb{C}$, if $f(z)f'(z)/z \ne 0$ and

$$\text{(1.3.4)} \qquad \operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0, \quad z \in U.$$

We denote the set of all α-convex functions by M_α. Clearly, $M_0 = S^*$ and $M_1 = CV$. By restricting α to be real, Miller et al. [81] have shown that if f is in M_α for $\alpha < 1$, then f is starlike, while if $\alpha \ge 1$, then f is convex. In fact, Sakaguchi had studied some of the properties of functions f satisfying (1.3.4) earlier in [120]. The notion of α-convexity has been further generalized in [6].

The following class generalizes the class $S^*(A,B)$.

Definition 1.3.6 A function $f \in \mathcal{A}_1$ is said to be in the class $SP^\lambda(A,B)$, $-1 \le B < A \le 1$, $-\pi/2 < \lambda < \pi/2$, if and only if,

$$\frac{1}{\cos\lambda}\left(e^{i\lambda}\frac{zf'}{f} - i\sin\lambda\right) \in P(A,B).$$

We have $SP^0(A,B) = S^*(A,B)$.

The class $SP^{\lambda}(1,-1)$ wider than the class S^*, was studied by Spacek [145]. For a function $f \in SP^{\lambda}(1,-1)$ geometrically it means that, for each w in $f(U)$ and $t \geq 0$ the logarithmic spiral $w \exp(e^{-i\lambda}t)$ is contained in $f(U)$. We call functions in $SP^{\lambda}(1,-1)$ to be λ-spirallike and denote the set of all such functions by $SP(\lambda)$. We have $SP(0) = S^*$ and $SP^{\lambda}(A,B) \subseteq SP(\lambda)$.

Libera [69] investigated the class $SP^{\lambda}(1-2\rho,-1)$, $0 \leq \rho < 1$ which is a generalization of $SP(\lambda)$. We say functions in $SP^{\lambda}(1-2\rho,-1)$ to be λ-spirallike of order ρ and denote it by $SP(\lambda,\rho)$. We have $SP(\lambda,\rho) \subseteq SP(\lambda)$, $SP(\lambda,0) = SP(\lambda)$ and, $SP(0,\rho)$ is contained in $S^*(\rho)$.

It is known [155] that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in SP^{\lambda}(A,B)$, then

$$|a_2| \leq (A-B)\cos\lambda,$$

$$|a_3| \leq \frac{(A-B)}{2}\cos\lambda\,\left|Be^{i\lambda} + (B-A)\cos\lambda\right|.$$

Silvia [140] introduced the following generalization of α-convex and λ-spirallike functions.

<u>Definition</u> 1.3.7 A function $f \in A_1$ is said to be α-λ spiral of order β, $\alpha \geq 0$, $|\lambda| < \pi/2$, λ real, $0 \leq \beta < 1$, if $f(z)f'(z)/z \neq 0$ for $z \in U$ and

$$\sec\lambda\left[(e^{i\lambda}-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] - i\tan\lambda \ll \frac{1+(1-2\beta)z}{1-z}.$$

We denote the class of all α-λ spiral functions of order β by $SP^{\lambda}_{\alpha}(\beta)$. We have $SP^{o}_{\alpha}(0) = M_{\alpha}$, $SP^{\lambda}_{o}(\beta) = SP(\lambda,\beta)$, $SP^{o}_{1}(\beta) = CV(\beta)$ and

$SP_0^0(\beta) = S^*(\beta)$, $0 \le \beta < 1$. Silvia in fact has shown [140] that $SP_\alpha^\lambda(\beta) \subseteq SP_0^\lambda(\beta)$. It is known [155] that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $SP_\alpha^\lambda(\beta)$, then

$$|a_2| \le \frac{2(1-\beta)\cos\lambda}{|\alpha + e^{i\lambda}|} .$$

$$|a_3| \le \frac{(1-\beta)\cos\lambda \; |(\alpha+e^{i\lambda})^2 + 2(1-\beta)\cos\lambda.(3\alpha+e^{i\lambda})|}{|2\alpha + e^{i\lambda}|\;|\alpha + e^{i\lambda}|^2} .$$

Another well known subclass of univalent functions is the class of close-to-convex functions introduced by Kaplan [60].

<u>Definition</u> 1.3.8 A function f in A_1 is said to be close-to-convex in U (or merely close-to-convex) if there exists a function h in CV such that

$$\text{(1.3.5)} \qquad \mathrm{Re}\left(\frac{f'(z)}{e^{i\beta}\,h'(z)}\right) > 0 , \qquad z \in U,$$

for some real β with $|\beta| < \pi/2$.

We denote the class of close-to-convex functions by CC. Loosely speaking, a function $f \in A$ is close-to-convex, if and only if, none of the curves $f(|z| = r)$, $0 < r < 1$, makes a "reverse hair pin turn". Precisely, the requirement is that as θ increases, the tangent direction $\arg\{(\partial/\partial\theta)\, f(re^{i\theta})\}$ should never decrease by as much as π from any previous value [60]. Ozaki [94] and Kaplan [60] proved that $CC \subseteq S$. Reade [107], [108] discovered that the Bieberbach conjecture holds for CC.

Choosing $h = f/z$, $\beta = 0$ in (1.3.5), we obtain that $S^* \subseteq CC$.

If we select $h(z) \equiv z$, $\beta = 0$ in (1.3.5), then we get the class R of functions f in CC that satisfy Re $f'(z) > 0$ in U. The class R was extensively studied by MacGregor [72],[73]. The radii for univalence and convexity of partial sums of functions in the class R or its subclasses were determined in [96].

A generalization of R is given as follows [29],[75]:

Definition 1.3.9 A function $f \in \mathcal{A}_1$ is said to be in the class $R(\alpha)$, $0 \le \alpha < 1$, if it satisfies that Re $f'(z) > \alpha$ in U.

Clearly $R(0) = R$ and $R(\beta) \subseteq R(\alpha)$ for $0 \le \alpha \le \beta < 1$.

It is known [88] that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R(\alpha)$, $0 \le \alpha < 1$, then

$$|a_n| \le 2(1-\alpha)/n\,, \qquad n = 2,3,\ldots$$

Given two analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in a disc $|z|<R_1$, $R_1 > 0$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in a disc $|z| < R_2$, $R_2 > 0$, their Hadamard product (or convolution) $f(z)*g(z)$ is defined by $f(z)*g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ and represents an analytic function in the disc $|z| < R_1R_2$. We denote $(f*g)(z) = f(z)*g(z)$.

The following basic result of Ruscheweyh and Sheil-Small [118] plays an important role in proving several convolution results.

Let $\phi \in CV$ and $g \in S^*$. Then for each $F \in \mathcal{A}$ and satisfying Re $F(z) > 0$ in U, we have

(1.3.6) $$\operatorname{Re} \frac{(\phi * gF)(z)}{(\phi * g)(z)} > 0, \qquad z \in U.$$

The conclusion of the above result implies that $(\phi * Fg)/(\phi * g)$ takes values only in the convex hull of the range of $F(z)$ over U for every $F \in \mathcal{A}$.

Ruscheweyh and Sheil-Small [118] also proved that if ϕ and g are in $S^*(1/2)$ and $F \in \mathcal{A}$ with $\operatorname{Re} F(z) > 0$ in U then $\frac{(\phi * gF)(z)}{(\phi * g)(z)}$ takes values in the convex hull of $F(U)$.

<u>Definition</u> 1.3.10 A function $f \in \mathcal{A}$ with $f(0) = 0$, $f'(0) \neq 0$ is said to be in the class $\mathcal{S}_\alpha$, $\alpha \leq 1$, if and only if,

$$\operatorname{Re}\, (zf'(z)/f(z)) \geq \alpha \qquad \text{in } U.$$

Suffridge [151] found that

(1.3.7) $$\begin{cases} \text{if } \alpha \leq \beta \leq 1 \text{ and } \sum\limits_{k=1}^{\infty} \gamma(\alpha,k)\, a_k z^k \in \mathcal{S}_\alpha \text{ where} \\ \gamma\,(t,k) \text{ are as in } z/(1-z)^{2(1-t)} = \sum\limits_{k=1}^{\infty} \gamma(t,k)\, z^k \\ \text{then } \sum\limits_{k=1}^{\infty} \gamma(\beta,k)\, a_k z^k \in \mathcal{S}_\beta. \end{cases}$$

An interesting class of prestarlike functions of order α was introduced by Ruscheweyh [116] as follows using convolution techniques.

<u>Definition</u> 1.3.11 A function $f \in \mathcal{A}$ with $f(0) = 0$, $f'(0) \neq 0$ is called prestarlike of order α, $\alpha \leq 1$, if and only if,

$$\left[\begin{array}{ll} \operatorname{Re} \dfrac{f(z)}{zf'(0)} > \dfrac{1}{2}, & z \in U, \quad \alpha = 1, \\ \dfrac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in \mathcal{S}_\alpha, & \alpha < 1. \end{array}\right.$$

We denote the class of prestarlike functions of order α by R_α.

Clearly $R_0 \cap A_1 = CV$ and $R_{1/2} \cap A_1 = S^*(1/2)$. Suffridge [151] and Ruscheweyh [116] showed that i) $R_\alpha \subseteq R_\beta$ for $\alpha \le \beta \le 1$, ii) if f, g $\in R_\alpha$, $\alpha \le 1$ then $f*g \in R_\alpha$. The class R_α may contain non-univalent functions for $1/2 < \alpha < 1$ [137]. A result analogous to that in (1.3.6) for R_α is due to Ruscheweyh [116]:

For $\alpha \le 1$, let $\phi \in R_\alpha$, $g \in S_\alpha$ and $F \in A$ with Re $F(z) > 0$ in U. Then for $z \in U$

$$\text{Re}\ \frac{(\phi*gF)(z)}{(\phi*g)(z)} > 0.$$

Sheil-Small et al. [129] further generalized the concept of prestarlike functions of order α.

Let $\phi = \phi(f,f',f'')$ and $\psi = \psi(f,f',f'')$, where f is in CV or $S^*(1/2)$, be such that Re $\phi > 0$ in U. Recently, Singh and Paul [143] studied the following types of problems:

(i) To find the largest number ρ, $0 < \rho < 1$ such that Re $(\phi + \psi) > 0$ in the disc $|z| < \rho$.

(ii) To find the ranges of scalars λ and μ such that Re $(\lambda\phi + \mu\psi) > 0$ in U.

Problem (i) is solved in [143] with $\phi = f/z$, $\psi = 1/f'$ and $\phi = z^2f''/f$, $\psi = zf'/f$ where f is in $S^*(1/2)$ and radii ρ are obtained as $\sqrt{4\sqrt{2}-5}$ and $\sqrt{8\sqrt{2}-11}$ respectively. For functions f in CV and $\phi = 1+zf''/f'$ and $\psi = 1/f'$ it is found [143] that

(1.3.8) the radius ρ in Problem (i) is $(\sqrt{5}-1)/\sqrt{2}$.

Further, Problem (ii) is investigated in [143] with $\phi = zf'/f$ and $\psi = s_n(z,f)/f$ where $s_n(z,f)$ denotes the n-th partial sum of $f(z)$ and f is in $S^*(1/2)$. Thus, it is determined that

$$L(z) = \text{Re}\ (\ \lambda \frac{zf'(z)}{f(z)} + \mu \frac{s_n(z,f)}{f(z)}\ )$$

is positive in U if (i) $\lambda \geq 0$, $\mu \geq 0$ and atleast one of them is nonzero (ii) $\mu \in \mathbb{C}$, $\lambda > 4|\mu|$. The result is sharp in the sense that the ranges of λ and μ can not be increased.

Problem (ii) is also studied in [143] with $\phi = f/zf'$ and $\psi = 1/f'$ when $f \in CV$. In fact, it is shown that Re $(\lambda\phi + \mu\psi) > 0$ in U for $\lambda > 2\mu \geq 0$.

1.4 The functions in the class S whose non-zero Taylor coefficients, from the second on, are negative have many interesting properties. Polynomials in S whose nonzero Taylor coefficients from the second on are negative were first investigated by Schild in [121]. Silverman [130] further extended this study to the infinite power series case.

<u>Definition</u> 1.4.1 A function $f \in A_1$ is said to be in the class T, if and only if, f(z) is univalent in U and is of the form

$$(1.4.1) \qquad f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0.$$

<u>Definition</u> 1.4.2 A function $f \in A_1$ and of the form (1.4.1) is said to be in the class C, if f(z) is convex and univalent in U.

Clearly $C \subseteq CV$. The classes T, C and their subclasses have been studied extensively by Silverman [130], Gupta and Jain [49], [50], Silverman and Silvia [138], Kapoor and Mishra [61] and others. Throughout in this section, unless otherwise stated, we assume that function f is of the form (1.4.1).

It is known [130] that, if $f \in T$, then

$$(1.4.2) \qquad a_n \leq 1/n, \qquad n = 2,3, \ldots .$$

The inequality (1.4.2) is sharp. A necessary and sufficient condition for a function f to be in T is that

$$(1.4.3) \qquad \sum_{n=2}^{\infty} n\, a_n \leq 1.$$

In view of the inequalities (1.2.6) and (1.4.3), the functions in T are starlike, i.e., T is contained in S^*. The classes C and T are closely related by an Alexander type relation:

$$(1.4.4) \qquad f \in C, \text{ if and only if, } zf' \in T.$$

It follows from (1.4.2) and (1.4.4) that if $f \in C$, then

$$(1.4.5) \qquad a_n \leq 1/n^2 , \qquad n = 2,3, \ldots$$

The inequality (1.4.5) is sharp. Similarly (1.4.3) and (1.4.4) give that for $f \in C$, it is necessary and sufficient that

$$\sum_{n=2}^{\infty} n^2 a_n \leq 1.$$

The following is a generalization [130] of the class T.

<u>Definition</u> 1.4.3 A function $f \in A_1$ of the form (1.4.1) is said to be in the class $T^*(\alpha)$, $0 \le \alpha \le 1$, if and only if,

$$\operatorname{Re}\,(zf'(z)/f(z)) \ge \alpha \qquad z \in U.$$

Clearly $T^*(0) = T$, $T^*(\alpha) \subseteq T^*(\beta)$ for $0 \le \beta \le \alpha \le 1$, $T^*(1) = \{z\}$ and $T^*(\alpha) \subseteq S^*(\alpha)$, $0 \le \alpha \le 1$. It is known [130] that for a function f to be in $T^*(\alpha)$, a necessary condition is

$$(1.4.6) \qquad a_n \le \frac{1-\alpha}{n-\alpha}, \qquad n = 2,3,\ \ldots$$

The inequality (1.4.6) is sharp. Silverman [130] also determined that a necessary and sufficient condition for f to be in $T^*(\alpha)$, $0 \le \alpha \le 1$ is

$$(1.4.7) \qquad \sum_{n=2}^{\infty} (n-\alpha)\, a_n \le 1-\alpha.$$

A generalization of the class C is the following class [130].

<u>Definition</u> 1.4.4 A function $f \in A_1$ of the form (1.4.1) is said to be in the class $C(\alpha)$, $0 \le \alpha \le 1$, if $\operatorname{Re}\,(1 + zf''(z)/f'(z)) \ge \alpha$ in U.

Clearly $C(0) = C$, $C(\alpha) \subseteq C(\beta)$ for $0 \le \beta \le \alpha \le 1$, $C(1) = \{z\}$ and $C(\alpha) \subseteq CV(\alpha)$ for $0 \le \alpha \le 1$. It is known [130] that $C(\alpha)$ is contained in $T^*(2/(3-\alpha))$, the number $2/(3-\alpha)$ is sharp for $0 \le \alpha \le 1$ and that $T^*(\alpha)$ is not contained in $C(\alpha)$ for $0 \le \alpha < 1$. An Alexander type relation exists between the classes $C(\alpha)$ and $T^*(\alpha)$: $f \in C(\alpha)$, if and only if, $zf' \in T^*(\alpha)$, $0 \le \alpha \le 1$.

Silverman [130] proved that if $f \in C(\alpha)$, then

(1.4.8) $$a_n \le \frac{1-\alpha}{n(n-\alpha)}, \qquad n = 2,3, \ldots$$

The inequality (1.4.8) is sharp. Further, $f \in C(\alpha)$, if and only if,

$$\sum_{n=2}^{\infty} n(n-\alpha)\, a_n \le 1-\alpha.$$

Sekine [124] introduced the following class $A(n,\{B_k\})$ which reduces to almost all classes of analytic functions with negative coefficients [124] for special choices of n and $\{B_k\}_{k=n+1}^{\infty}$.

<u>Definition</u> 1.4.5 A function $f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{A}_1$, $a_k \ge 0$, $k \ge n+1$, $n = 1,2,3,\ldots,$ is said to be in the class $A(n,\{B_k\})$, $B_k > 0$ for $k \ge n+1$, if $\sum_{k=n+1}^{\infty} B_k a_k \le 1$.

The class $A(2,\{B_k\})$ was introduced by Silverman in [134].

It is known [124] that if $f \in A(n,\{B_k\})$ and $B_k \le B_{k+1}$, then the growth bounds are given by, for $z \in U$,

$$\max\,(0,\ r-r^{n+1}/B_{n+1}) \le |f(z)| \le r+r^{n+1}/B_{n+1}, \quad |z| = r.$$

The following class is a subclass of $A(2,\{B_k\})$, for $B_k \equiv b_k$ and was introduced in [3].

<u>Definition</u> 1.4.6 A function $f(z) = z - (p/b_2)z^2 - \sum_{n=3}^{\infty} a_n z^n$ $(0 \le p \le 1,\ a_n \ge 0)$ is said to be in the class $F_p(\{b_n\})$ if there exists a sequence $\{b_n\}_{n=2}^{\infty}$ of positive real numbers such that

$$p + \sum_{n=3}^{\infty} b_n\, a_n \le 1.$$

Several subclasses of T consisting of functions with a fixed

second coefficient follow as special cases of $F_p(\{b_n\})$. Thus $F_p(\{(n-\alpha)/(1-\alpha)\})$ and $F_p(\{n(n-\alpha)/(1-\alpha)\})$ were studied in [58].

Ahuja and Silverman [3] obtained that the radius for convexity in $F_p(\{b_n\})$ is the largest value of r for which

$$\frac{4p}{b_2} r + \frac{n^2(1-p)r^{n-1}}{b_n} \leq 1, \qquad (n = 3,4, \ldots).$$

Let $V(\theta_n;\beta)$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A_1$ such that for $a_n \neq 0$, $\theta_n + (n-1)\beta \equiv \pi(\text{mod } 2\pi)$, β real where $\arg a_n = \theta_n$.

<u>Definition</u> 1.4.7 A function $f \in A_1$ is said to be in the class $SV(\theta_n;\beta)$, β real if, $f \in V(\theta_n;\beta)$ is univalent in U.

The union of $SV(\theta_n;\beta)$ taken over all possible sequences $\{\theta_n\}_{n=2}^{\infty}$ and all real numbers β is denoted by SV. Silverman [133] determined that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in SV$, then

$$\sum_{n=2}^{\infty} n|a_n| \leq 1. \tag{1.4.9}$$

The subclass of $SV(\theta_n;\beta)$ consisting of functions starlike of order α, $0 \leq \alpha \leq 1$ is denoted by $SV^*(\theta_n;\beta)$ and the subclass of SV consisting of functions starlike of order α, $0 \leq \alpha \leq 1$ is denoted by $SV^*(\alpha)$. It is known [133] that, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in SV^*(\alpha)$, $0 \leq \alpha \leq 1$, then

$$\sum_{n=2}^{\infty} (n-\alpha) |a_n| \leq 1-\alpha. \tag{1.4.10}$$

We denote by $\mathcal{B}_\alpha$ and R_α, $0 \le \alpha < 1$, the subclass of $V = \bigcup_{\{\theta_n\},\beta\in\mathbb{R}} V(\theta_n;\beta)$ consisting of functions in $\mathcal{B}(\alpha)$ and $R(\alpha)$ respectively. Srivastava and Owa [146] found that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{B}_\alpha$, $0 \le \alpha < 1$, then

$$\sum_{n=2}^{\infty} |a_n| \le 1-\alpha \tag{1.4.11}$$

and if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R_\alpha$, then

$$\sum_{n=2}^{\infty} n|a_n| \le 1-\alpha. \tag{1.4.12}$$

Silverman [131] introduced the following class of functions with two fixed points.

<u>Definition</u> 1.4.8 A function $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, $a_1 > 0$, $a_n \ge 0$ for $n \ge 2$ is said to be in the class $S_o^*(\alpha,z_o)$, $0 \le \alpha < 1$, $-1 < z_o < 1$, $z_o \ne 0$, if it satisfies that $f(z_o) = z_o$ and

$$\operatorname{Re}\,(zf'(z)/f(z)) > \alpha \quad \text{in } U.$$

It is observed [131] that functions in $S_o^*(\alpha,z_o)$ are univalent. A necessary and sufficient condition for $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ with $a_1 = 1 + \sum_{n=2}^{\infty} a_n z_o^{n-1} > 0$, $a_n \ge 0$ for $n \ge 2$, to be in $S_o^*(\alpha,z_o)$ is [131] that

$$\sum_{n=2}^{\infty} \left(\frac{n-\alpha}{1-\alpha} - z_o^{n-1} \right) a_n \le 1. \tag{1.4.13}$$

The study of $S_o^*(\alpha,z_o)$ has been further extended in [84].

A class closely related to $S_o^*(\alpha, z_o)$ is the following [131]:

<u>Definition</u> 1.4.9 A function $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, $a_1 > 0$, $a_n \geq 0$ for $n \geq 2$ is said to be in the class $K_o(\alpha, z_o)$, $0 \leq \alpha < 1$, $-1 < z_o < 1$, $z_o \neq 0$, if it satisfies $f(z_o) = z_o$ and

$$\text{Re } (1+zf''(z)/f'(z)) > \alpha \qquad \text{in } U.$$

It is known [131] that $K_o(\alpha, z_o) \subseteq S_o^*(2/(3-\alpha), z_o)$ and thus consists of univalent functions only. A necessary and sufficient condition for $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ with $a_1 = 1 + \sum_{n=2}^{\infty} a_n z_o^{n-1} > 0$, $a_n \geq 0$ for $n \geq 2$, to be in $K_o(\alpha, z_o)$ is due to Silverman [131]:

$$\sum_{n=2}^{\infty} \left(\frac{n(n-\alpha)}{1-\alpha} - z_o^{n-1} \right) a_n \leq 1. \tag{1.4.14}$$

The results of Silverman [131] for $S_o^*(\alpha, z_o)$, $K_o(\alpha, z_o)$ were extended to p-valent cases in [84].

Lakshma Reddy and Padmanabhan [68] further generalized the classes $S_o^*(\alpha, z_o)$ and $K_o(\alpha, z_o)$ as follows:

<u>Definition</u> 1.4.10 A function $f(z) = a_1 z - \sum_{n=1}^{\infty} a_{n+k} z^{n+k} \in \mathcal{A}$, $k \geq 1$, $a_1 > 0$, $a_{n+k} \geq 0$, is said to be in the class $S_1(A, B, z_o)$, $-1 < z_o < 1$, $z_o \neq 0$, $-1 \leq A < B \leq 1$, if $f(z_o) = z_o$ and

$$zf'(z)/f(z) \ll (1+Az)/(1+Bz).$$

We have $S_1(2\alpha-1, 1, z_o) = S_o^*(\alpha, z_o)$ and that [68] functions in $S_1(A, B, z_o)$ are univalent in U. It is known that [68] for $f(z) = a_1 z - \sum_{n=1}^{\infty} a_{n+k} z^{n+k}$, $k \geq 1$, $a_1 > 0$, $a_{n+k} \geq 0$, $f(z_o) = z_o$,

$-1 < z_o < 1$, we have $f \in S_1(A,B,z_o)$, if and only if,

$$(1.4.15) \qquad \sum_{m=k+1}^{\infty} (m(B+1) - (A+1) - (B-A)\, z_o^{m-1})\, a_m \le B-A.$$

<u>Definition</u> 1.4.11 A function $f(z) = a_1 z - \sum_{n=1}^{\infty} a_{n+k} z^{n+k} \in \mathcal{A}$, $k \ge 1$, $a_1 > 0, a_{n+k} \ge 0$, is said to be in the class $K_1(A,B,z_o)$, $-1 < z_o < 1$, $z_o \ne 0$, $-1 \le A < B \le 1$ if $f(z_o) = z_o$ and

$$1+zf''(z)/f'(z) \ll (1+Az)/(1+Bz).$$

We have $K_1(2\alpha-1,1,z_o) = K_o(\alpha,z_o)$ and that $K_1(A,B,z_o)$ consists of univalent functions only [68]. It is known [68] that $f(z) = a_1 z - \sum_{n=1}^{\infty} a_{n+k} z^{n+k} \in \mathcal{A}$, $k \ge 1$, $a_1 > 0$, $a_{n+k} \ge 0$, $f(z_o) = z_o$, we have $f \in K_1(A,B,z_o)$, if and only if,

$$(1.4.16) \qquad \sum_{m=k+1}^{\infty} (m(m(B+1) - (A+1)) - (B-A)\, z_o^{m-1})\, a_m \le B-A.$$

Another generalization of the classes $S_o^*(\alpha,z_o)$ and $K_o(\alpha,z_o)$ due to Mishra and Sahu [85] is as follows:

<u>Definition</u> 1.4.12 Let $s(z) = \sum_{k=1}^{\infty} c_k z^k \in \mathcal{A}$ with $c_1 > 0$, $c_k \ge 0$, $k \ge 2$ and $g(z) = \sum_{k=1}^{\infty} d_k z^k \in \mathcal{A}$ with $d_1 > 0$, $d_k \ge 0$, $k \ge 2$, $(c_k/c_1) - (d_k/d_1) > 0$, $k = 2,3, \ldots$, $0 \le \alpha \le c_1/d_1$. A function $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ with $a_k \ge 0$, $k = 1,2, \ldots$, is said to be in the class $F[s,g,\alpha,z_o]$, z_o real, $0 < |z_o| < 1$, if $f(z_o) = z_o$, $(g*f)(z) \ne 0$, in the annulus $0 < |z| < 1$, and

$$\mathrm{Re}\, \frac{(s*f)(z)}{(g*f)(z)} > \alpha \qquad \text{in } U.$$

The classes

$$F\,[\,z(1-z)^{-2},\ z(1-z)^{-1},\ \alpha,\ z_o],$$

$$F\,[z(1+z)(1-z)^{-3},\ z(1-z)^{-2},\ \alpha,\ z_o]$$

correspond to the classes $S_o^*(\alpha,z_o)$ and $K_o(\alpha,z_o)$ of Silverman [131]. Mishra and Sahu [85] found that for $s(z) = \sum_{k=1}^{\infty} c_k z^k \in A$ with $c_1 > 0$, $c_k \geq 0$, $k \geq 2$ and $g(z) = \sum_{k=1}^{\infty} d_k z^k \in A$ with $d_1 > 0$, $d_k \geq 0$, $k \geq 2$, $(c_k/c_1) - (d_k/d_1) > 0$, $k = 2,3, \ldots$, $0 \leq \alpha \leq c_1/d_1$, z_o real, $0 < |z_o| < 1$, the function $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \in A$ with $a_k \geq 0$, $k = 1,2, \ldots$, is in the class $F[s,g,\alpha,z_o]$, if and only if,

$$(1.4.17) \qquad \sum_{k=2}^{\infty} \left(\frac{c_k - \alpha d_k}{c_1 - \alpha d_1} - z_o^{k-1} \right) a_k \leq 1.$$

For open problems on T or related classes, one may refer to [136].

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < R$, $0 < R \leq \infty$. Let $\{d_n\}_{n=1}^{\infty}$ denote a non-decreasing sequence of positive numbers and let D be the operator which transforms the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ into

$$Df(z) = \sum_{n=1}^{\infty} d_n\, a_n z^{n-1} .$$

The operator D, called the Gelfond-Leontev derivative was introduced and studied by Gelfond and Leontev [32] in connection with the generalization of Fourier series.

Juneja and Shah [59] initiated the study o univalence of successive Gelfond-Leontev derivatives. Patel [9 extended the study of univalence of Gelfond-Leontev derivati s of analytic functions.

Definition 1.4.13 A function $f \in T$ is said to be in the class $T_1(D)$, if and only if, its Gelfond-Leontev deriv tive $Df(z)$ is analytic and univalent in U.

For $d_n \equiv n$, we denote $T_1(D)$ by T_1. Silverman studied T_1 in [135]. Patel [98] has shown that in general, the class T_1 need not be contained in $T_1(D)$ and conversely $T_1(D)$ also need not be contained in T_1. The clas $T_1(D)$, being a subclass of T, the inequality $a_2 \le 1/2$ continues to hold even for functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T_1(D)$. That this inequality is sharp can be seen by considering the function $\rho(z) = z - z^2/2 \in T_1(D)$. A bound on a_3 is found in [98], for functions f in $T_1(D)$ with restricted a_2. Thus,

$$(1.4.18)\qquad \begin{cases} \text{if } f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T_1(D),\ \beta_0 \equiv \sup_{n\ge 2} \{d_n/n\} \text{ and} \\ \\ 0 < a_2 \le \beta_0/d_2 \quad \text{then} \quad a_3 < \beta_0/2d_3. \end{cases}$$

A sufficient condition for $f \in T$ with $a_2 > 0$ to be in $T_1(D)$ is that [98]

$$(1.4.19)\qquad \sum_{n=3}^{\infty} (n-1)\, d_n a_n \le d_2 a_2.$$

It is also proved in [98] that if $0 < a_2 \le 2d_3/(4d_3+3d_2)$ and (1.4.19) holds then $f \in T_1(D)$.

For $d_n \equiv n$, conditions (1.4.18) and (1.4.19) reduce to the analogous conditions for T_1 found earlier by Silverman [135].

Definition 1.4.14 A function $f \in T$ is said to be in the class $C_1(D)$ if f is convex and its Gelfond-Leontev derivative $Df(z)$ is analytic, univalent and convex in U.

Let C_1 denote the class of functions in T for which f' is also convex in U. We observe that $C_1(D) \subseteq C$ and, for $d_n \equiv n$, $C_1(D)$ reduces to C_1. In general, [98] the class C_1 need not be contained in $C_1(D)$ and conversely $C_1(D)$ need not be contained in C_1. Patel [98] determined a sufficient condition for a function f in C with $a_2 > 0$ to be in $C_1(D)$ as follows:

(1.4.20) $$\sum_{n=3}^{\infty} (n-1)^2 d_n a_n \le d_2 a_2.$$

It is known [98] that if $0 < a_2 \le 4d_3/(16d_3 + 9d_2)$ and (1.4.20) holds then $f \in C_1(D)$.

1.5 The problem of finding necessary and sufficient conditions for polynomials to belong to one of the following classes has been a subject of investigation by various workers for nearly four decades.

Patel [98] has found both necessary and sufficient conditions for a trinomial to be in $T_1(D)$ or $C_1(D)$. Thus

$$(1.5.1)\begin{cases} f(z) = z-a_2z^2-a_{p+1}z^{p+1},\ p \ge 2,\ (a_2 > 0,\ a_{p+1} \ge 0) \text{ is in} \\ T_1(D), \text{ if and only if,} \\ a_{p+1} \le \min \begin{cases} (1-2a_2)/(p+1) \\ d_2a_2/pd_{p+1}. \end{cases} \end{cases}$$

For $d_n \equiv n$, (1.5.1) reduces to the analogous condition for T_1 found earlier by Silverman [135]. Further,

$$(1.5.2)\begin{cases} f(z) = z-a_2z^2-a_{p+1}z^{p+1},\ p \ge 2,\ (a_2 > 0,\ a_{p+1} \ge 0) \text{ is in} \\ C_1(D), \text{ if and only if,} \\ a_{p+1} \le \min \begin{cases} (1-4a_2)/(p+1)^2 \\ d_2a_2/p^2d_{p+1}. \end{cases} \end{cases}$$

<u>Definition</u> 1.5.1 Let for $p \ge 2$,

$S^*_{2p-1} = \{f(z) = z+a_pz^p + a_{2p-1}z^{2p-1} \in S^*:\ a_p,\ a_{2p-1} \text{ real}\}$,

$St_3 = \{f(z) = z+a_2z^2 + a_3z^3 \in S$: f has radius for starlikeness unity; a_2, a_3 real$\}$,

$CV_{2p-1} = \{f \in CV:\ zf' \in S^*_{2p-1}\}$,

$S_4 = \{f(z) = z+a_2z^2 + a_3z^3 + z^4/4 \in S:\ a_2, a_3 \text{ real}\}$,

$S_5 = \{f(z) = z+\alpha_3z^3 + tz^5 \in S:\ \alpha_3 \text{ real},\ t \ge 0\}$.

Clearly $St_3 \subseteq S^*_3$. Ruscheweyh and Wirths [119] found necessary and sufficient conditions for $\rho(z) = z+a_pz^p + a_{2p-1}z^{2p-1}$ to be in S^*_{2p-1}, $p \ge 2$. Thus

$p \in S^{*}_{2p-1}$, with a_p, a_{2p-1} real, if and only if,

$$(1.5.3)\quad |a_p| \le \begin{cases} \dfrac{1+(2p-1)a_{2p-1}}{p}, & -\dfrac{1}{2p-1} \le a_{2p-1} \le \dfrac{p+1}{(2p-1)(3p-1)} \\[2ex] 4\left(\dfrac{(1-(2p-1)a_{2p-1})pa_{2p-1}}{(p+1)^2-(3p-1)^2 a_{2p-1}}\right)^{1/2}, & \dfrac{p+1}{(2p-1)(3p-1)} \le a_{2p-1} \le \dfrac{1}{2p-1} . \end{cases}$$

Brannan and Brickman [17] obtained earlier necessary and sufficient conditions for $p_3(z) = z+a_2z^2+a_3z^3$ to be in St_3. Thus they showed that $p_3 \in St_3$ with a_2, a_3 real, if and only if,

$$(1.5.4)\quad \begin{cases} a_2 = \pm\dfrac{1+3a_3}{2}, & -\dfrac{1}{3} \le a_3 \le \dfrac{1}{5} \\[2ex] a_2^{\,2} = 32a_3\,(1-3a_3)(9-25a_3)^{-1}, & \dfrac{1}{5} \le a_3 \le \dfrac{1}{3} . \end{cases}$$

Necessary and sufficient conditions under which $p_4(z) = z+a_2z^2 + a_3z^3 + z^4/4$ is in S_4 are due to Brannan [16]. Thus he found that p_4 is in S_4 with a_2, a_3 real, if and only if,

$$3a_3 = 2a_2, \qquad \frac{-3(\sqrt{5}+1)}{8} \le a_2 \le \frac{3(\sqrt{5}+1)}{8} .$$

Kossler [66], Brannan [16], Cowling and Royster [23] and Rahman and Waniurski [105] determined necessary and sufficient conditions for trinomials of particular form to be univalent in U. Kasten [62] studied close-to-convex trinomials.

Rahman and Szynal [104] obtained necessary and sufficient conditions under which $p_5(z) = z+\alpha_3 z^3 + tz^5$ is in S_5. Thus they showed that $p_5 \in S_5$ with α_3 real, if and only if,

$$|\alpha_3| \le \begin{cases} \dfrac{1+5t}{3}, & 0 \le t \le \dfrac{1}{10} \\ 2\sqrt{t(1-t)} - t, & \dfrac{1}{10} \le t \le \dfrac{1}{5} \end{cases} \tag{1.5.5}$$

A survey of univalent polynomials is given in [152].

Recently Ganeshan [31] has obtained necessary and sufficient conditions for a normalized cubic polynomial to be in $S^*(\alpha)$.

Let

D_1 = the closure of the domain bounded by $(x+1/3)^2 = y-2/9$, $(x-1/3)^2 = y-2/9$ and $x^2 = y+1/5$,

D_2 = the closure of the domain lying in $x \ge 0$ bounded by $x = y+1/5$ and $4x^2-3x-8xy+4y+4y^2 = 0$,

$D_3 = D_2 \cap \{(x,y) \in \mathbb{R}^2 : x \ge 0,\ y \ge x-1/3\}$.

A function $f \in \mathcal{A}$ is said to be typically real in U, if $f(z)$ is real in U only for reals. It is proved in [139] that for the function $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, $z \in U$, with b_1 and b_2 real $\psi(g) = z/g(z)$ is a typically real cubic polynomial in U, if and only if,

$$b_1 b_{n-1} - b_n = (b_1^2 - b_2)\, b_{n-2}, \qquad n = 3,4, \ldots \tag{1.5.6}$$

$$\text{and either } (b_1, b_2) \in D_1 \text{ or } (b_1^2, b_2) \in D_2. \tag{1.5.7}$$

Further, it is shown [139] that $\psi(g)$ is a univalent cubic polynomial, if and only if, (1.5.6) holds and

(1.5.8) $$(b_1, b_2) \in D_1 \text{ or } (b_1^2, b_2) \in D_3.$$

1.6 A function $f \in A_1$ with $f(z) \neq 0$ in $U\backslash\{0\}$ may be expressed as $f(z) = z/g(z)$ where $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \in A$. We denote $\psi(g) = z/g(z)$. Thus for $z \in U$,

(1.6.1) $$f(z) = \psi(g) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}.$$

Such a function g is uniquely determined and we call it the reciprocal of f in U.

The study of properties of f vis-a-vis Taylor coefficients of its reciprocal function g is interesting. Prawitz [103] determined the following necessary condition in terms of the Taylor coefficients b_n's of $g(z)$ when $f = \psi(g)$ is in S. Thus, he proved that if $\psi(g) \in S$, then

(1.6.2) $$\sum_{n=2}^{\infty} (n-1) |b_n|^2 \leq 1.$$

Mitrinovic [86] studied the converse problem, i.e., a sufficient condition in terms of the Taylor coefficients b_n's of $g(z)$ for $\psi(g)$ to be in S. He determined the following sufficient condition in [86]. Thus, if

(1.6.3) $$|b_1| + \sum_{n=2}^{\infty} (n-1) |b_n| \leq 1$$

then $\psi(g) \in S$.

Reade et al. [109] showed that the sufficient condition (1.6.3) in fact implies that $\psi(g) \in S^*$. Sufficient conditions in terms of the Taylor coefficients b_n's are also determined [109] for the class $S^*(\alpha)$, $0 \le \alpha < 1$ and CV as follows. Thus, if

$$(1.6.4) \qquad \sum_{n=2}^{\infty} (n-1+\alpha)\,|b_n| \le \begin{cases} (1-\alpha) - (1-\alpha)\,|b_1|, & 0 \le \alpha \le 1/2 \\ (1-\alpha) - \alpha\,|b_1|, & 1/2 \le \alpha \le 1 \end{cases}$$

then $\psi(g) \in S^*(\alpha)$. Further, if

$$(1.6.5) \qquad 4\,|b_1| + \sum_{n=2}^{\infty} (n-1)\,(3n+1)\,|b_n| \le 1$$

then $\psi(g) \in CV$.

The study of reciprocal functions has been further extended in [110]. Ahuja and Jain [2] obtained sufficient conditions in terms of Taylor coefficients b_n's of $g(z)$ for $\psi(g)$ to be in $SP(\lambda,\rho)$: If ρ, λ are constants, $0 \le \rho < 1$, $-\pi/2 < \lambda < \pi/2$ and

$$(1.6.6) \qquad \sum_{k=1}^{\infty} [k+\{k^2 - 4(1-\rho)(k+\rho-1)\cos^2\lambda\}^{1/2}]\,|b_k| \le 2\,(1-\rho)\cos\lambda$$

then $\psi(g) \in SP(\lambda,\rho)$.

For $\lambda = 0$, condition (1.6.6) reduces to condition (1.6.4).

Robertson [112] generalized the concept of convex functions of order α as follows:

<u>Definition</u> 1.6.1 A function $f \in \mathcal{A}_1$ is said to be a λ-Robertson function of order α, $-\pi/2 < \lambda < \pi/2$, $0 \le \alpha < 1$ if it satisfies $\mathrm{Re}\,[e^{i\lambda}(1+zf''(z)/f'(z))] > \alpha \cos\lambda$ in the unit disc U.

We denote the set of λ-Robertson functions of order α in U by $C^{\lambda}(\alpha)$. Thus, $C^{0}(\alpha) = CV(\alpha)$.

Sufficient condition in terms of b_n's for $\psi(g)$ to be a λ-Robertson function of order α in U is also determined in [2]. This condition generalizes condition (1.6.5). Thus, if $0 \le \alpha < 1$, $-\pi/2 < \lambda < \pi/2$, $\psi(g) \in \mathcal{A}_1$ with

$$\frac{3+(1-\alpha)\cos\lambda}{(1-\alpha)\cos\lambda}\,|b_1| + \sum_{n=2}^{\infty} \frac{(n-1)(3n+(1-\alpha)\cos\lambda)}{(1-\alpha)\cos\lambda}\,|b_n| \le 1,$$

then $\psi(g) \in C^{\lambda}(\alpha)$.

Recently, Silverman and Silvia [139] found necessary condition in terms of the Taylor coefficients b_n's of $g(z)$ when $\psi(g)$ is in $T^*(\alpha)$. Thus, if $\psi(g) \in T^*(\alpha)$, then

$$(1.6.7) \qquad |b_n| \le \frac{1-\alpha}{n+1-\alpha}, \qquad n = 0,1,2, \ldots$$

The inequality (1.6.7) is sharp for $g_n(z) = 1 + \sum_{k=1}^{\infty} \left(\frac{1-\alpha}{n+1-\alpha} z^n\right)^k$, $\psi(g_n) = z - ((1-\alpha)/(n+1-\alpha))\, z^{n+1}$.

1.7 Let X be a complex linear space with a topology defined on it. If the vector addition defined on X x X and the scalar multiplication defined on $\mathbb{C}$ x X, are continuous on the product spaces, then X is said to be a linear topological space. A subset Y of X is said to be convex if $tx_1 + (1-t)\,x_2$ belongs to Y for all

distinct x_1 and x_2 in Y, $0 < t < 1$. A linear topological space is said to be locally convex if every neighbourhood of the additive identity 0 in X contains a convex neighbourhood of 0. It is well known [154] that the class $\mathcal{A}$ is a localy convex linear topological space with respect to the topology of uniform convergence on compact subsets of U.

Let Y be a subset of a complex linear topological space. The convex hull of Y is the smallest convex set containing Y and is denoted by Co(Y).

<u>Definition</u> 1.7.1 Let Y be a subset of a complex linear topological space X. The <u>closed</u> <u>convex</u> <u>hull</u> of Y is the intersection of all closed convex sets containing Y.

We denote the closed convex hull of Y by $\overline{Co}(Y)$.

<u>Definition</u> 1.7.2 Let Y be a subset of a complex linear space X. A point ξ in Y is called an extreme point of Y if $\xi \neq tx_1+(1-t)x_2$ for any distinct x_1, x_2 in Y and $0 < t < 1$.

We denote the set of extreme points of Y by Ext{Y}.

The following important theorem is helpful in determining extreme points of a subset Y of a linear topological space X.

<u>Krein-Mil'man</u> <u>Theorem</u> [42] Let C be a compact convex set in a locally convex linear topological space X. Then C is the closed convex hull of its extreme points, i.e., $C = \overline{Co}(\text{Ext}\{C\})$.

Thus it follows that if C is a nonempty convex subset of $\mathcal{A}$, then Ext{C} is also nonempty.

Extreme points of closed convex hulls of well known families S, S^*, CV and CC were determined by Brickman et al. [19]. Thus

$$\text{(1.7.1)} \qquad \operatorname{Ext}\{\overline{\mathrm{Co}}(S^*)\} = \{ \frac{z}{(1-xz)^2} : |x| = 1 \}.$$

The extreme points of closed convex hulls of $S^*(\alpha)$ and $CV(\alpha)$ were determined in [20].

Some extremal problems for univalent functions can be stated in terms of continuous linear functionals on $\mathcal{A}$. The following characterization of continuous linear functionals due to Toeplitz [156] is much useful.

(1.7.2) Each continuous linear functional on $\mathcal{A}$ has the form

$$L(h) = \sum_{n=0}^{\infty} a_n b_n, \quad h(z) = \sum_{n=0}^{\infty} a_n z^n$$

for some sequence $\{b_n\}$ of complex constants such that

$\limsup_{n\to\infty} |b_n|^{1/n} < 1$. Conversely, each such sequence $\{b_n\}$ generates a continuous linear functional L in this manner.

<u>Definition</u> 1.7.3 Let X be a complex linear topological space and $Y \subseteq X$. A point $x \in Y$ is called a support point of Y if there exists a continuous linear functional L, non-constant on Y, such

that $\mathrm{Re}(L(x)) \geq \mathrm{Re}(L(y))$ for all $y \in Y$.

We denote the set of support points of Y by Supp{Y}.

It follows [26] from the Krein-Mil'man Theorem that there is atleast one extreme point among the support points associated with every continuous linear functional, i.e., if Y is a subset of a locally convex linear topological space X and a continuous linear functional J on X has a support point $x \in Y$ with Re{J} non-constant on Y, then there is an extreme point y of Y such that $\mathrm{Re}\{J(y)\} = \mathrm{Re}\{J(x)\}$.

Deeb [25] determined the set of support points of T:

$$\mathrm{Supp}\{T\} = \{\ f \in T:\ f(z) = z - \sum_{n=2}^{\infty} \lambda_n z^n/n\ ,\ \lambda_n \geq 0,\ \sum_{n=2}^{\infty} \lambda_n \leq 1,\ \lambda_j = 0 \text{ for some } j\ \}.$$

<u>Definition</u> 1.7.4 Let $P^*(\alpha,\beta,\mu)$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $0 \leq \mu \leq 1$ be the class of functions $f(z) = z - \sum_{n=2}^{\infty} |a_n|\ z^n \in \mathcal{A}$ for which

$$\left| \frac{f'(z)-1}{\mu f'(z) + 1-(1+\mu)\alpha} \right| < \beta \qquad \text{in } U.$$

It is known [92] that a function $f(z) = z - \sum_{n=2}^{\infty} |a_n|\ z^n$ is in $P^*(\alpha,\beta,\mu)$, if and only if,

$$(1.7.3) \qquad \sum_{n=2}^{\infty} n(1+\mu\beta)|a_n| \leq (1+\mu)\ \beta(1-\alpha).$$

The set of support points of $P^*(\alpha,\beta,\mu)$ is also determined in [92].

Owa et al. found the support points of $T^*(\alpha)$ and $C(\alpha)$, $0 \leq \alpha < 1$ in [93].

The linear convexity technique in univalent function theory is discussed in detail in [52].

1.8 In this section a brief introduction of the thesis is given. The thesis consists of seven chapters.

Chapter I is introduction and consists of basic definitions and known results used in the subsequent chapters of the thesis.

Chapter II is devoted to the study of the class $CVG(R_1,R_2)$, $0 \le R_1 \le R_2 \le \infty$, $R_2 \ge 1$ recently introduced by Styer and Wright (cf. Definition 1.2.6, [150]). First, certain necessary conditions in terms of $d^* = \sup_{\zeta \in \partial U} |f(\zeta)|$ for f to be in $CVG(R_1,R_2)$ are determined in this chapter. One of these conditions gives a lower bound on d^* if $f \in CVG(R_1,R_2)$. Necessary and sufficient conditions are determined for R_1 to be equal to R_2 if f is in $CVG(R_1,R_2)$. Further in this chapter, the results of following nature are obtained for the class $CVG(R_1,R_2)$: (i) growth bounds on $|f(z)|$ in terms of d^* (ii) lower growth bound on $|f(z)|$ involving $|z|$ and R_1 in the disc $|z| < R_o$, $R_o \cong 0.543$ (iii) bounds on the functional $|(f(z_1) - f(z_2))/(z_1-z_2)|$ for certain distinct z_1, z_2 in the unit disc U (iv) distortion bounds on $|f'(z)|$ in the disc $|z| \le 3 - \sqrt{8}$ (v) a rotation theorem when $R_2 < \infty$ and (vi) an estimate of Euclidean curvature $k(f;z)$. Finally, an open problem of Goodman [44] is solved for $CVG(R_1,R_2)$, $R_2 < \infty$ in this chapter by showing that $CVG(R_1,R_2) \subseteq CV(\gamma)$ where $\gamma = 2R_2-1-2\sqrt{R_2^2 - R_2}$ and

γ is the largest possible such a number.

In Chapter III the concept of α-curvature, $\alpha < 1$ is introduced for functions analytic and locally univalent in the unit disc U and the resulting classes $CV_\alpha(R_1,R_2), 0 \le R_1 \le R_2 \le \infty$, $0 \le \alpha < 1$ and $C_\alpha(K)$, $K > 0$, $\alpha < 1$ are studied. For $0<R_1 \le R_2 < \infty$, functions in $CV_\alpha(R_1,R_2)$ are called convex functions of bounded α-type. While some of the results contained in the chapter for the classes $CV_\alpha(R_1,R_2)$ and $C_\alpha(K)$ are analogues of the results of Goodman [44], [46] for the class $CV(R_1,R_2) \equiv CV_0(R_1,R_2)$ (cf. Definition 1.2.5) and those of Wirths [159], [160] for the class $C(K) \equiv C_0(K)$, (cf.Definition 1.2.7) several other results found in the chapter are new. First, the sharp lower bound on γ so that a function in $CV_\alpha(R_1,R_2)$, $R_2 < \infty$ is convex of order γ is obtained. Then an integral operator that transforms convex functions of bounded α-type into convex functions of bounded type [44] is studied. This integral operator is helpful in finding distortion bounds for the class $CV_\alpha(R_1,R_2)$. The other results obtained in this chapter for the class $CV_\alpha(R_1,R_2)$ are of the following nature: (i) growth theorem on $|f(z)|$ (ii) bounds on the functional $|(f(z_1) - f(z_2))/(z_1-z_2)|$ for certain distinct z_1,z_2 in U and (iii) a rotation theorem when $R_2 < \infty$. Next, necessary and sufficient conditions for a function f to be in the class $C_\alpha(K)$ are found. The other results for the class $C_\alpha(K)$ found in this chapter include (i) the bounds on the second and third Taylor

series coefficients for functions in $C_\alpha(K)$ and (ii) distortion bounds.

A function f analytic in the unit disc U, normalized by $f(0) = 0$, $f'(0) = 1$ with $f(z) \neq 0$ in the punctured disc $U\setminus\{0\}$, may be expressed as

$$(1.8.1) \qquad f(z) = \psi(g) = z/g(z) \text{ in } U,$$

where

$$(1.8.2) \qquad g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \text{ in } U.$$

For $\psi(g)$ varying over a certain class of polynomials of degree atmost n ($n \geq 2$), the reciprocal coefficient region is defined in Chapter IV. First, the reciprocal coefficient regions of certain classes of univalent polynomials are determined in this chapter. Further, necessary and sufficient conditions in terms of the reciprocal coefficient regions are found for a polynomial $\psi(g)$ to be in certain subclasses of starlike functions or convex functions or in certain other known subclasses of univalent functions. The reciprocal coefficient regions of certain classes of univalent polynomials determined in this work are improvements over the reciprocal coefficient region found by Silverman and Silvia [139] for certain bigger class of univalent polynomials.

In Chapter V, necessary and sufficient conditions in terms of the coefficients b_n are determined for the function $\psi(g)$ to be in certain classes of analytic functions. Necessary and

sufficient conditions in terms of the reciprocal coefficient regions for a trinomial $\psi(g)$ to be in certain subclasses of univalent functions with univalent Gelfond-Leontev derivatives in U are determined. The bounds on b_n, $1 \le n \le 4$ are found, when a trinomial $\psi(g)$ is univalent and has univalent Gelfond-Leontev derivative in U. Necessary as well as sufficient conditions in terms of $\{b_n\}_{n=1}^{\infty}$ are determined for the function $\psi(g)$ to be in each one of certain subclasses of starlike functions or spirallike functions. Some of the results in this chapter generalize the results of Reade et al. [109].

In Chapter VI the support points, growth theorems and distortion theorems for certain new subclasses of analytic functions are determined. Among the new classes considered are (i) the class $A\{n,M_k\}$, $n = 1,2,3,\ldots,$ consisting of functions $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ in U with $\arg a_k = -\arg M_k$ for $a_k \neq 0$ and $\sum_{k=n+1}^{\infty} M_k a_k \le 1$ where $\{M_k\}_{k=n+1}^{\infty}$ is a sequence of nonzero complex numbers and (ii) the class $A_o(n,B_k,z_o)$, $n = 1,2,3,\ldots,$ z_o real, $0 < |z_o| < 1$, $B_k > 0$, consisting of functions with two fixed points 0, z_o. The extreme points of each of the classes $A\{n,M_k\}$ and $A_o(n,B_k,z_o)$ are determined. The support points of certain subclasses of univalent functions with univalent Gelfond-Leontev derivatives in U and the class $A\{n,M_k\}$ with $|M_k| \ge k$ are described. The support points of the class $A_o(n,B_k,z_o)$ with $B_k \ge k$ are also determined. Growth and

CHAPTER II

CONVEX FUNCTIONS AND ROLLING CIRCLES CRITERION

2.1 In this chapter certain new results are developed for Goodman's class $CV(R_1,R_2)$ of convex functions of bounded type (cf. Definition 1.2.5, [44]) and the class $CVG(R_1,R_2)$ of Styer and Wright (cf. Definition 1.2.6, [150]). These results complement the existing results in the literature for the classes $CV(R_1,R_2)$ and $CVG(R_1,R_2)$.

For a function f in the class $CV(R_1,R_2)$, Goodman [44] obtained bounds for d and d^*, where d and d^* are respectively the distances of the nearmost and the farthermost points on $\partial f(U)$ from the origin. Thus, he proved

$$R_2 - \sqrt{R_2^2 - R_2} \le d \le R_1 - \sqrt{R_1^2 - R_1} \tag{2.1.1}$$

and

$$R_1 \le \frac{d^2}{2d-1} \le R_2 \tag{2.1.2}$$

where the right hand side inequality in (2.1.1) and the left hand side inequality in (2.1.2) hold if $R_1 \ge 1$. Further,

$$(2.1.3) \qquad d^* \le R_2 + \sqrt{R_2^2 - R_2}.$$

Styer and Wright [150] observed that the inequalities (2.1.1) and (2.1.3) continue to hold for the class $CVG(R_1,R_2)$. The method of proof of the inequality (2.1.2) in [44] shows that this inequality also holds for the class $CVG(R_1,R_2)$ and is sharp. The inequalities (2.1.1) and (2.1.2) are necessary conditions on R_1 and R_2 in terms of $d = d(f)$ for a function f to be in the class $CVG(R_1,R_2)$. However, an analogue of these conditions in terms of d^* is not known. Further, (2.1.3) gives only the upper bound on d^* and a lower bound on d^* has not been determined so far.

In this chapter, Section 2.2 is aimed at the determination of analogues of the necessary conditions (2.1.1) and (2.1.2) involving d^* in place of d, for the functions in the class $CVG(R_1,R_2)$. A lower bound on $d^* = d^*(f)$ is also obtained in this section when $f \in CVG(R_1,R_2)$. Finally in this section, necessary and sufficient conditions are determined for R_1 to be equal to R_2, if the function f is in the class $CVG(R_1,R_2)$.

For $f \in CV(R_1,R_2)$, Goodman [44], [46] found that

$$(2.1.4) \qquad |f(z)| \le 2R_2-d$$

and

$$(2.1.5) \qquad |f(z)| \le \frac{rd(2R_2-d)}{R_2(1-r)+rd}$$

in the disc $|z| = r \le 1$ where $d = \inf_{\zeta\in\partial f(U)} |\zeta|$. Both the

inequalities are sharp. His proof shows that the inequality (2.1.4) continues to hold for the class $CVG(R_1, R_2)$ also.

In [46], Goodman also showed that, if $f \in CV(R_1, R_2)$, then

$$(2.1.6) \qquad |f(z)| \le r \frac{R_2}{R_2 - r\sqrt{R_2^2 - R_2}}$$

for $|z| = r \in [r^*(d), 1)$ where $r^*(d) = 2R_2(R_2-d)/(2R_2(R_2-d)+d^2)$ and the inequality is sharp.

The upper bounds in the inequalities (2.1.4) and (2.1.5) involve d. The analogues of these inequalities, involving $d^* = \sup\limits_{\zeta \in \partial f(U)} |\zeta|$, in place of d have not yet been found so far for the class $CV(R_1, R_2)$ or $CVG(R_1, R_2)$.

In Section 2.3 analogues of the inequalities (2.1.4) and (2.1.5) involving d^* are found for the class $CVG(R_1, R_2)$. An analogue of the inequality (2.1.6) for the functions in the class $CVG(R_1, R_2)$ is also found in this section, wherein the number $r^*(d)$ is replaced by a number r^* which is independent of d. In the same section, a lower bound on $|f(z)|$ is also determined for z in the disc $|z| < R_0 \cong 0.543$, and functions f in the class $CVG(R_1, R_2)$. Finally, in this section bounds of the functional $|(f(z_1) - f(z_2))/(z_1 - z_2)|$ for certain distinct z_1, z_2 in U and $f \in CVG(R_1, R_2)$ are obtained.

For $f \in CV(R_1,R_2)$, Goodman [46] found that

$$(2.1.7) \qquad |f'(z)| \le \frac{R_2}{1-r^2}$$

in the disc $|z| = r < 1$. The function F_R, defined by

$$F_R(z) = \frac{z}{1 - \sqrt{1-1/R}\, z}, \quad R \ge 1, \; z \in U,$$

shows that the inequality (2.1.7) is sharp for each $r \in (0,1)$, and $R = R_2 = 1/(1-r^2)$. It may also be observed that the inequality (2.1.7) is a consequence of (1.2.3) and (1.2.36).

From the proof of the inequality (2.1.7), it can be observed that the inequality continues to hold for the class $CVG(R_1,R_2)$. However, an analogue of the inequality (2.1.7) for the class $CVG(R_1,R_2)$ in terms of $d^* = \sup_{\zeta \in \partial f(U)} |\zeta|$ is not known. In Section 2.4, a result in this direction is found for the class $CVG(R_1,R_2)$. A lower bound on $|f'(z)|$ for z in the disc $|z| \le 3 - \sqrt{8}$ and a rotation theorem are also derived in this section for the functions $f \in CVG(R_1,R_2)$. Finally, in this section, a sharp ν such that $CVG(R_1,R_2)$, $R_2 < \infty$ is contained in $CV(\nu)$ is found.

2.2 For a function $f \in CVG(R_1,R_2)$ (cf.Definition 1.2.6), we first find some relations between the smallest and the largest distances of the image curve $\partial f(U)$ from the origin. Let,

$$d = d(f) = \inf_{\zeta \in \partial f(U)} |\zeta| \text{ and } d^* = d^*(f) = \sup_{\zeta \in \partial f(U)} |\zeta|.$$

In the following propositions an analogue of the inequality (2.1.2) in terms d^* is found.

Proposition 2.2.1. If $f \in CVG(R_1, R_2)$, then

$$1 \le \frac{(d^*)^2}{2d^*-1} \le R_2. \tag{2.2.1}$$

The inequalities in (2.2.1) are sharp for the function F_{R_2}, defined by

$$F_{R_2}(z) = \frac{z}{1 - \sqrt{1 - 1/R_2}\, z}, \qquad R_2 \ge 1,\ z \in U. \tag{2.2.2}$$

Proof. Let

$$G(x) = \frac{x^2}{2x-1}$$

for $x \in (1/2, \infty)$. This function is increasing in x if $1 \le x < \infty$. Therefore, for $1 \le d^* < \infty$, the inequality (2.2.1) follows from the inequality (2.1.3). The function $G(x)$ is decreasing in x if $1/2 < x < 1$. Therefore, for $1/2 < d \le d^* < 1$, the inequality (2.2.1) follows from (2.1.2) since $d \le d^*$. In the case $d^* = \infty$, inequality (2.2.1) follows from Definition 1.2.6. In the remaining case $1/2 = d \le d^* < 1$, the inequality (2.1.2) gives that $R_2 = \infty$ and thus the inequality (2.2.1) follows. •

The function F_{R_2}, given by (2.2.2), is in the class $CVG(R_2,R_2) \subseteq CVG(R_1,R_2)$ with $d^* = 1/(1 - \sqrt{1-1/R_2})$ so that

$$\frac{(d^*)^2}{2d^*-1} = R_2,$$

which gives the sharpness of the inequality (2.2.1).

<u>Remark</u>. For $f \in CVG(R_1,R_2)$, the inequality (2.2.1) gives a better lower bound on R_2 than that of inequality (2.1.2), if $d/(2d-1) \le d^*$. There does exist a function in the class $CVG(R_1,R_2)$ satisfying $d/(2d-1) < d^*$; consider for example, $f_2(z) = 2\log(1-z/2)^{-1} \in \overline{CV}(1,2/\sqrt{3})$. For the function F_{R_2}, given by (2.2.2), $d/(2d-1) = d^*$.

<u>Proposition</u> 2.2.2. If $f \in CVG(R_1,R_2)$ with $R_1 \ge 1$, then

$$\frac{(d^*)^2}{2d^*-1} \ge R_1 \tag{2.2.3}$$

and

$$d^* \ge R_1 + \sqrt{R_1^2 - R_1}. \tag{2.2.4}$$

The inequalities (2.2.3) and (2.2.4) are sharp when $R_1 = R_2$.

<u>Proof</u>. Let $d^* < \infty$ and $f(e^{it_o}) \in \partial f(U)$ be such that $d^* = |f(e^{it_o})|$, for some real t_o. By making a suitable rotation of f, it may be taken that $f(e^{it_o}) = -d^*$. Then the unit exterior normal to $\partial f(U)$ at $f(e^{it_o})$ is $n(e^{it_o}) = -1$. And by Definition 1.2.6, we have

$$D(R_1 - d^*, R_1) \subseteq f(U),$$

where $D(a,R)$ denotes the disc $|z - a| < R$, $a \in \mathbb{C}$, $R > 0$.

Equivalently,

$$\frac{Bz}{1-Az} \ll f(z)$$

where $B = (2R_1-d^*)d^*/R_1$ and $A = (R_1-d^*)/R_1$. This implies (cf.(1.3.1))

$$B \leq 1$$

or

$$\frac{(d^*)^2}{2d^*-1} \geq R_1$$

which is the inequality (2.2.3). When $d^* = \infty$, the inequality (2.2.3) follows directly.

Inequality (2.2.3) gives that

$$(d^*-(R_1 + \sqrt{R_1^2 - R_1}))\ (d^* - (R_1 - \sqrt{R_1^2 - R_1})) \geq 0.$$

By Definition 1.2.6 we have

$$d^* \geq R_1.$$

These last two inequalities, together, give the inequality (2.2.4).

The function F_{R_2}, given by (2.2.2) is in the class $CVG(R_2,R_2) \subseteq CVG(R_1,R_2)$ with $d = 1/(1+a)$, $d^* = 1/(1-a)$, where $a = \sqrt{1-1/R_2}$ so that

$$\frac{(d^*)^2}{2d^*-1} = R_2$$

and

$$d^* = R_2 + \sqrt{R_2^2 - R_2} \; ,$$

thereby giving the sharpness of the inequalities (2.2.3) and (2.2.4).

Corollary 2.2.1. If $f \in CVG(R_1,R_2)$, then

$$R_1 \le \frac{(d^*)^2}{2d^*-1} \le R_2. \tag{2.2.5}$$

Proof. Proposition 2.2.1 and the inequality (2.2.3), together give the corollary.

Remark. For $f \in CVG(R_1,R_2)$ with $R_1 \ge 1$, by solving a quadratic equation in d^*, it can be seen that the inequality (2.2.3) gives better upper bound for R_1 than that given by inequality (2.1.2), if and only if, $d^* \le d/(2d-1)$. There does exist a function in the class $CVG(R_1,R_2)$ with $R_1 \ge 1$, satisfying $d^* < d/(2d-1)$; consider, for example, $f_3(z) = e^z-1 \in \overline{CV}(1,\infty)$ for $0 < \operatorname{Im} z \le 2\pi$.

The following result gives a necessary and sufficient condition for R_1 to be equal to R_2, if $f \in CVG(R_1,R_2)$.

Theorem 2.2.1 Let $f \in CVG(R_1,R_2)$ with $1 \le R_1 \le R_2 < \infty$. Then,

$$R_1 = R_2 \; ,$$

if and only if,

(i) $d^* = R_2 + \sqrt{R_2^2 - R_2}$

(ii) $d + d^* = R_1 + R_2$

(iii) $d = R_2 - \sqrt{R_2^2 - R_2}$

<u>and</u>

(iv) f <u>maps</u> U <u>onto a disc of radius</u> R_2.

<u>Proof</u>. Let $R_1 = R_2$. Then (i) follows from the inequalities (2.1.3) and (2.2.4).

The inequality (2.1.1) gives (iii).

(ii) follows from (i) and (iii). (iv) follows from Definition 1.2.6 and the assumption. This completes the proof of 'only if' part.

Conversely, let (i), (ii), (iii) and (iv) hold. (i) and (iii) together give

$$d + d^* = 2R_2.$$

This, together with (ii) gives that $R_1 = R_2$.
Thus, the proof of the theorem is complete.

2.3 In this section the properties of the growth of $|f(z)|$ are studied when $f \in CVG(R_1,R_2)$ (cf. Definition 1.2.6).

Goodman's [44] proof of the inequality (2.1.4) continues to hold for the class $CVG(R_1,R_2)$ also. However analogues of the inequalities (2.1.4) and (2.1.5), involving $d^* = \sup_{\zeta \in \partial f(U)} |\zeta|$, are

not known. In this section these analogues are derived.

This section also consists of an analogue of the inequality (2.1.6) for the functions in the class $CVG(R_1, R_2)$ wherein the number r^* is independent of d.

In the following proposition, an analogue of the inequality (2.1.4) involving d^* in place of d is found. In Theorem 2.3.1, an improvement of this proposition in the unit disc will be obtained.

<u>Proposition</u> 2.3.1. If $f \in CVG(R_1, R_2)$ with $R_2 < \infty$, then

$$|f(z)| \le r(R_2 + |R_2 - d^*|) \tag{2.3.1}$$

in the disc $|z| = r \le 1$. The inequality is sharp for $R_1 = R_2$.

<u>Proof</u>. From the definition of d^*, we have that

$$|f(z)| \le d^*$$

in the disc $|z| = r \le 1$. By the triangle inequality,

$$|f(z)| \le R_2 + |R_2 - d^*|.$$

Now Schwarz lemma gives the inequality (2.3.1).

For the function F_{R_2} given by (2.2.2), $R_1 = R_2$,

$$d^* = R_2 + \sqrt{R_2^2 - R_2} \ge R_2$$

$$|F_{R_2}(1)| = \frac{1}{1 - \sqrt{1 - 1/R_2}} = d^*$$

$$= R_2 + |R_2 - d^*|.$$

Thus, the sharpness of the inequality (2.3.1) follows.

Corollary 2.3.1. If $f \in CVG(R_1,R_2)$ with $d^* \le R_2 < \infty$, then

$$|f(z)| \le r(2R_2 - d^*)$$

in the disc $|z| = r \le 1$.

Proof. The inequality in the corollary is straight forward in view of the inequality (2.3.1).

Remarks (i) Corollary 2.3.1 improves Goodman's [44] result given by the inequality (2.1.4).

(ii) The functions f in the class $CVG(R_1,R_2)$ satisfying $d < d^* < R_2 < \infty$ do exist as can be seen from the following example. For integer $k \ge 2$ and $0 < a < 1/k^2$, the binomial $p_{a,k}(z) = z + az^k \in CVG(R_1,R_2)$ with $R_2 = (1-ka)^2/(1-k^2a)$. Further, for $p_{a,k}(z)$, $d = 1-a < d^* = 1+a$, so that for $k = 2$, $d^* < R_2$ for $1/8 < a < 1/4$ and for $k \ge 3$, $d^* < R_2$ for $0 < a < 1/k^2$.

(iii) An analogue of the inequality in Corollary 2.3.1 involving R_1 can also be found. Thus, if $f \in CVG(R_1,R_2)$ with $R_2 < \infty$, then

$$|f(z)| \le r(R_1 + |R_1 - d^*|) = rd^* \tag{2.3.2}$$

in the disc $|z| = r \le 1$. This inequality is sharp for $R_1 = R_2$, the sharpness function being F_{R_2}, given by (2.2.2).

The upper bound of $|f(z)|$ given by the inequality (2.3.2) is

better than that given by the inequality (2.3.1).

Theorem 2.3.1. *If* $f \in CVG(R_1, R_2)$ *with* $0 < R_1 \le R_2 < \infty$, *then*

(2.3.3) $$\frac{rd^*|2R_1-d^*|}{R_1(1-r)+rd^*} \le |f(z)| \le \frac{rd^*(2R_2-d^*)}{R_2-|R_2-d^*|r}$$

where $|z| = r$, *the left hand side inequality holds in the disc* $|z| < R_o \cong 0.543$, *where* R_o *is the least positive root of*

$$\text{Arc}\sin x + 2\,\text{Arc}\tan x = \pi/2,$$

and the right hand side inequality holds in the disc $|z| \le 1$. *Both the inequalities are sharp for* $R_1 = R_2$.

Proof. By making a suitable rotation of f we may obtain that $f(e^{it_o}) = -d^* = -\sup\limits_{\zeta\in\partial f(U)} |\zeta|$, for some t_o real. We have the unit exterior normal to $\partial f(U)$ at $f(e^{it_o})$, $n(e^{it_o}) = -1$. Now, by Definition 1.2.6, we get that

$$f(U) \subseteq D(R_2 - d^*, R_2)$$

or

$$f(z) \ll \frac{Bz}{1-Az}$$

where $B = d^*(2R_2-d^*)/R_2$ and $A = (R_2-d^*)/R_2$.

The inverse of the function $g(z) = Bz/(1-Az)$ is $h(z) = z/(Az+B)$ and the function $j(z) = (h\circ f)(z)$ satisfies the conditions of Schwarz lemma. So

$$|f(z)| \le r(|Af(z)| + B)$$

in the disc $|z| = r \le 1$. This implies that

$$|f(z)| \le \frac{rB}{1-r|A|} . \tag{2.3.4}$$

By substituting the values of A and B in this, the right hand side inequality of (2.3.3) is obtained.

Now, to prove the left hand side inequality in (2.3.3), we apply Definition 1.2.6 and obtain

$$\frac{B^* z}{1-A^* z} \ll f(z)$$

where $B^* = d^*(2R_1-d^*)/R_1$ and $A^* = (R_1-d^*)/R_1$.

Further,

$$\left|\frac{B^* z}{1-A^* z}\right| \ge \frac{|B^*|r}{1+|A^*|r}$$
$$= \frac{rd^*|2R_1-d^*|}{R_1+(d^*-R_1)r}$$

in the disc $|z| = r < 1$.

Hence, by relation (1.3.2) we have that

$$\frac{rd^*|2R_1-d^*|}{R_1(1-r)+rd^*} \le \left|\frac{B^* z}{1-A^* z}\right| \le |f(z)|$$

in the disc $|z| < R_o$, where R_o is as in the statement of the theorem. This gives the left hand side inequality of (2.3.3).

The function F_{R_2}, given by (2.2.2), is in the class

$CVG(R_2,R_2) \subseteq CVG(R_1,R_2)$. For this function, $d^* = 1/(1-a) \geq R_2$ so that

$$\frac{rd^*(2R_2-d^*)}{R_2-|R_2-d^*|r} = \frac{r}{1-ar} = |F_{R_2}(r)|$$

for $r \in [0,1)$ in U and

$$\frac{rd^*|2R_1-d^*|}{R_1(1-r)+rd^*} = \frac{r}{1+ar} = |F_{R_2}(-r)|$$

for $r \in [0,R_o)$, where $a = \sqrt{1-1/R_2}$ and thus equality is attained in inequality (2.3.3).

Remarks. (i) For $f \in CVG(R_1,R_2)$ with $R_2 < \infty$ and $r = 1$, the upper bound of $|f(z)|$ in the inequality (2.3.3) and that given by the inequality (2.3.1) are equal. For $r < 1$, the upper bound given by the inequality (2.3.3) is better than that given by the inequality (2.3.1).

(ii) From the proof of Theorem 2.3.1, it can be observed that inequality (2.3.3) with d^* replaced by d everywhere, continues to remain true and sharp, i.e., if $f \in CVG(R_1,R_2)$ with $0 < R_1 \leq R_2 < \infty$, then

$$\frac{rd|2R_1-d|}{R_1+|R_1-d|r} \leq |f(z)| \leq \frac{rd(2R_2-d)}{R_2(1-r)+rd} \tag{2.3.5}$$

where $|z| = r$, the left hand side inequality holds in the disc $|z| < R_o$, where R_o is as in Theorem 2.3.1 and the right hand side inequality holds in the disc $|z| \leq 1$.

The function F_{R_2} given by (2.2.2), is in the class $CVG(R_2,R_2) \subseteq CVG(R_1,R_2)$. For this function $d = 1/(1+a) \leq R_2$ so that

$$\frac{rd(2R_2-d)}{R_2(1-r)+rd} = \frac{r}{1-ar} = |F_{R_2}(r)|$$

for $r \in [0,1)$ and

$$\frac{rd|2R_1-d|}{R_1+|R_1-d|r} = \frac{r}{1+ar} = |F_{R_2}(-r)|$$

for $r \in [0,R_0)$, where $a = \sqrt{1-1/R_2}$ and thus equality is attained in the inequality (2.3.5).

(iii) Let

$$Q(r,R_2,x) = \frac{x(2R_2-x)}{R_2-|R_2-x|r} .$$

It can be seen that for $r \in [r^*,1)$, the function $Q(r,R_2,x)$ is decreasing in x for $x \leq R_2$ and hence the upper bound of $|f(z)|$ in the inequality (2.3.3) is better than that in the inequality (2.3.5), for $R_2 \geq d^*$ where

$$r^* = \frac{2\sqrt{R_2^2 - R_2}}{2R_2-1} .$$

(iv) Let

$$P(r,R_1,x) = \frac{x|2R_1-x|}{R_1+|R_1-x|r} .$$

It can be seen that for $r \in [0,R_o)$ and R_o as in Theorem 2.3.1, the function $P(r,R_1,x)$ is decreasing in x for $x \in [R_1,2R_1]$ and hence the lower bound of $|f(z)|$ in the inequality (2.3.5) is better than that in the inequlity (2.3.3) for $R_1 \le d \le d^* \le 2R_1$; the last inequality does hold for the function

$$P_{a,3}(z) = z + az^3 \in CVG((1+3a)^2/(1+9a),R_2),$$

where, $0 \le a \le 1/15$.

(v) For $f \in \overline{CV}(R_1,R_2)$ with $R_1 < R_2$, strict inequality holds in the right hand side inequality of (2.3.3), because, when equality holds, the inequality (2.3.4) gives that

$$f(z) = \frac{Cz}{1-Dz}$$

where,

$$C = \frac{e^{it'} d^*(2R_2-d^*)}{R_2} \text{ and } D = \frac{e^{it'}(R_2-d^*)}{R_2} ,$$

t' real, so that f has $R_1 = R_2$.

For $f \in CVG(R_1,R_2)$, the upper bound of $|f(z)|$ in the inequality (2.3.3) (or (2.3.5)) is dependent on d^* (or d). The following theorem gives an upper bound of $|f(z)|$ that is independent of both d and d^*.

Theorem 2.3.2. *If* $f \in CVG(R_1,R_2)$ *with* $1 \le R_1 \le R_2 < \infty$, *then*

$$(2.3.6) \qquad \frac{r(4R_1-1)}{2(2R_1+(2R_1-1)r)} \le |f(z)| \le \frac{rR_2}{R_2-r\sqrt{R_2^2-R_2}}$$

where, the left hand side inequality holds for $|z| = r \in [0,R_o)$, R_o *being as in Theorem 2.3.1 and the right hand side inequality holds for* $r \in [r^*,1]$; $r^* = 2\sqrt{R_2^2 - R_2}\,/(2R_2-1)$. *The right hand side inequality in (2.3.6) is sharp for* $R_1 = R_2$.

Proof. Set,

$$Q(r,R_2,d) = \frac{d(2R_2-d)}{R_2(1-r)+rd} .$$

Then, $r\,Q(r,R_2,d)$ is the upper bound of $|f(z)|$ in the inequality (2.3.5). Let

$$r^* = \frac{2\sqrt{R_2^2 - R_2}}{2R_2-1} .$$

For $r \in [r^*,1]$, the function $rQ(r,R_2,d)$ is decreasing in d. By the inequality (2.1.1), we have

$$d \ge R_2 - \sqrt{R_2^2 - R_2} .$$

Hence, for $r \in [r^*,1]$, we may replace d by $R_2 - \sqrt{R_2^2 - R_2}$ in $rQ(r,R_2,d)$ and obtain the right hand side inequality in (2.3.6) from the inequality (2.3.5).

Next, we prove the left hand side inequality of (2.3.6). The

lower bound of $|f(z)|$ in the inequality (2.3.5) is an increasing function in d for $d \in [R_2 - \sqrt{R_2^2 - R_2}, R_1 - \sqrt{R_1^2 - R_1}]$. Hence we may replace d by $R_2 - \sqrt{R_2^2 - R_2}$ in the left hand side inequality of (2.3.5) and obtain for z in the disc $|z| < R_o$,

$$|f(z)| \geq \frac{r(R_2 - \sqrt{R_2^2-R_2})(2R_1-R_2 + \sqrt{R_2^2-R_2})}{R_1 + (R_1-R_2 + \sqrt{R_2^2-R_2})r} .$$

The right hand expression in this inequality is a decreasing function of R_2 and its limit as R_2 tends to ∞ is $r(4R_1-1)/2(2R_1+(2R_1-1)r)$. Hence, the left hand side inequality of (2.3.6) follows from the last inequality.

The function F_{R_2}, given by (2.2.2), is in the class $CVG(R_2,R_2) \subseteq CVG(R_1,R_2)$ and

$$|F_{R_2}(r)| = \frac{rR_2}{R_2-r\sqrt{R_2^2 - R_2}}$$

for $r \in [r^*,1]$, which gives the sharpness in the right hand side inequality of (2.3.6).

<u>Remarks</u>. (i) For $f \in CVG(R_2,R_2)$, the upper bound of $|f(z)|$ in the inequality (2.3.6) is better than that in the inequality (2.3.1). Indeed, for the function

$$Q^*(r,R_2) = \frac{rR_2}{R_2-r\sqrt{R_2^2 - R_2}}$$

we have,

$$Q^*(r,R_2) \le \frac{rR_2}{R_2 - \sqrt{R_2^2 - R_2}}$$

$$= r(R_2 + \sqrt{R_2^2 - R_2})$$

$$= r(R_2 + |R_2 - d^*|)$$

by Theorem 2.2.1, for $r \in [r^*,1]$ where r^* is as in Theorem 2.3.2.

(ii) If $f \in \overline{CV}(R_1,R_2)$ and equality holds in the upper inequality of(2.3.6), then as in Remark (v) following Proof of Theorem 2.3.1, we obtain that $R_1 = R_2$. Hence strict inequality holds in the upper inequality of (2.3.6) when $R_1 < R_2$.

(iii) The right hand side inequality in (2.3.6) holds for any R_1, $0 \le R_1 \le R_2 < \infty$ and the left hand side inequality in (2.3.6) holds for any R_2 with $1 \le R_1 \le R_2 \le \infty$.

(iv) While proving the lower bound on $|f(z)|$ in the inequality (2.3.6), we obtained that for $f \in CVG(R_1,R_2)$, $1 \le R_1 \le R_2 < \infty$, for z in the disc $|z| < R_0$,

$$|f(z)| \ge \frac{r(R_2 - \sqrt{R_2^2-R_2})(2R_1-R_2 + \sqrt{R_2^2-R_2})}{R_1 + (R_1-R_2 + \sqrt{R_2^2-R_2})r},$$

where, R_0 is as in Theorem 2.3.1. For $R_1 = R_2$, the lower bound in this inequality is same as that in (1.2.42) found by Ma et al.[71].

Next an upper bound of the functional $\left|\frac{f(z_1) - f(z_2)}{z_1-z_2}\right|$ is found when $f \in CVG(R_1,R_2)$ for certain $z_1,z_2 \in U$, $|z_1| < |z_2|$.

<u>Theorem</u> 2.3.3. <u>If</u> $f \in CVG(R_1,R_2)$, <u>then</u>

$$\left|\frac{f(z_1) - f(z_2)}{z_1-z_2}\right| \le \frac{R_2}{2(|z_2| - |z_1|)} \ln \frac{(1+|z_2|)(1-|z_1|)}{(1-|z_2|)(1+|z_1|)} \tag{2.3.7}$$

<u>for</u> z_1,z_2 <u>in</u> U, $|z_1| < |z_2|$ <u>such that</u> $|z|$ <u>either increases or decreases on the line segment joining</u> z_1 <u>and</u> z_2.

<u>Proof</u>. We employ the integral method of Bieberbach [12] to derive the inequality (2.3.7). Let $L(z_1,z_2)$ be the segment $z=z(s)$, $0 \le s \le s'$ in U, $z(0) = z_1$, $z(s') = z_2$ joining the points z_1 and z_2 in U, $|z_1| < |z_2|$, such that $|z(s)|$ either increases or decreases on $L(z_1,z_2)$ and $\overline{L}(z_1,z_2)$ be its length, where s is the arc length. Let $L(z_1,z_2,f)$ be the image of $L(z_1,z_2)$ under $f(z)$ and $\overline{L}(z_1,z_2,f)$ be its length. Thus,

$$\overline{L}(z_1,z_2,f) = \int_0^{s'} |f'(z)|\ |z'(s)|ds$$

$$= \int_0^{s'} |f'(z)|\ |dz|$$

$$\overline{L}(z_1,z_2,f) \le \int_0^{s'} \frac{R_2}{1-|z|^2} |dz| \tag{2.3.8}$$

by the inequality (2.1.7). Further, $|z(s)|$ being increasing as s traverses from 0 to s', we have,

$$\int_0^{s'} \frac{R_2}{1-|z|^2}\,|dz| \le \frac{\mathcal{L}(z_1,z_2)}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \frac{R_2}{1-|z|^2}\,d|z|$$

$$= \frac{R_2\,\mathcal{L}(z_1,z_2)}{2(|z_2|-|z_1|)} \ln \frac{(1+|z_2|)(1-|z_1|)}{(1-|z_2|)(1+|z_1|)} .$$

Using the inequality (2.3.8), it follows that

$$\text{(2.3.9)} \qquad \mathcal{L}(z_1,z_2,f) \le \frac{R_2\,\mathcal{L}(z_1,z_2)}{2(|z_2|-|z_1|)} \ln \frac{(1+|z_2|)(1-|z_1|)}{(1-|z_2|)(1+|z_1|)} .$$

Moreover, we have

$$\left|\frac{f(z_1) - f(z_2)}{z_1-z_2}\right| \le \frac{\mathcal{L}(z_1,z_2,f)}{|z_1-z_2|}$$

$$= \frac{\mathcal{L}(z_1,z_2,f)}{\mathcal{L}(z_1,z_2)}$$

$$\le \frac{R_2}{2(|z_2|-|z_1|)} \ln \frac{(1+|z_2|)(1-|z_1|)}{(1-|z_2|)(1+|z_1|)}$$

by the inequality (2.3.9). This gives the required inequality.

Remark. For $f \in CVG(R_1,R_2)$ with $R_2 < \infty$, the upper bound of $|f(z)|$ obtained from the inequality (2.3.7) is better than Goodman's inequality (2.1.4) in the annulus $0 < |z| \le (e^2-1)/(e^2+1)$. In fact, for these special z, and $d = \min_{\zeta\in\partial f(U)} |\zeta|$,

$$|f(z)| \le R_2 \le 2R_2 - d.$$

Theorem 2.3.4. *If* $f \in CVG(R_1, R_2)$, $R_2 < \infty$, *then*

$$(2.3.10) \qquad \left|\frac{f(z_1) - f(z_2)}{z_1 - z_2}\right| \geq \frac{1}{(1+a|z_1|)(1+a|z_2|)}$$

for $z_1, z_2 \in U$, $|z_1| < |z_2|$. *Further, if* $|z|$ *either increases or decreases on the line segment joining* z_1, z_2; $|z_1| < |z_2| < 1$, *then*

$$(2.3.11) \qquad \left|\frac{f(z_1) - f(z_2)}{z_1 - z_2}\right| \leq \begin{cases} \dfrac{1}{(1-a|z_1|)(1-a|z_2|)}, & |z_2| \leq r(a) \\[2ex] \dfrac{1}{r_2 - r_1} \ln \dfrac{U(r_2, a)}{U(r_1, a)}, & r(a) < |z_1| \end{cases}$$

where

$$U(t,a) = t^{\frac{1}{2(1-a)}}(2+(1-a)t)^{\frac{1-a}{2(1+a)(3-a)}}(1+t)^{-R_2/2}(1-t)^{\frac{-1}{2(3-a)(1-a)}},$$

$r(a) = 2/(1+\sqrt{5-4a})$ *and* $a = \sqrt{1 - 1/R_2}$. *The inequalities* (2.3.10) for $z_1, z_2 \in U$ *and* (2.3.11) *for* z_1, z_2 *in the disc* $|z| \leq r(a)$ *are sharp.*

Proof. Let z_1, z_2 be in U such that $|z_1| < |z_2|$. Choose the curve $L(z_1, z_2) = z = z(s)$, $0 \leq s \leq s'$, $z_1 = z(0)$, $z_2 = z(s')$ in such a way that the image $L(z_1, z_2, f)$ of $L(z_1, z_2)$ under $f(z)$ is the line segment joining $f(z_1)$ with $f(z_2)$. It is enough to assume that $r'(s) = |z(s)|' \geq 0$ in the interval $|z_1| \leq s \leq |z_2|$. Let $\overline{L}(z_1, z_2)$

and $\overline{L}(z_1,z_2,f)$ be the lengths of $L(z_1,z_2)$ and $L(z_1,z_2,f)$ respectively. Thus

$$\overline{L}(z_1,z_2,f) = \int_0^{s'} |f'(z)|\ |z'(s)|ds = \int_0^{s'} |f'(z)|\ |dz|.$$

Let $\phi(|z|) = 1/(1+a|z|)^2$ where $a = \sqrt{1-1/R_2}$. It follows from the lower distortion bound in (1.2.44) for $CVG(R_1,R_2)$ that

$$\overline{L}(z_1,z_2,f) \geq \int_0^{s'} \phi(|z|)\ |dz|$$

$$\geq \frac{\overline{L}(z_1,z_2)}{|z_2|-|z_1|} \int_{|z_1|}^{|z_2|} \phi(|z|)\ d|z|$$

$$= \overline{L}(z_1,z_2) \frac{1}{(1+a|z_1|)(1+a|z_2|)}$$

which gives,

$$\left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \geq \frac{\overline{L}(z_1,z_2,f)}{\overline{L}(z_1,z_2)} \geq \frac{1}{(1+a|z_1|)(1+a|z_2|)} .$$

Thus, the inequality (2.3.10) is proved.

The proof of the inequality (2.3.11) is similar to that of the inequality (2.3.7) except that the upper distortion bounds in (1.2.44), (1.2.45) have to be used in place of (2.1.7). Details of proof are omitted. The function F_{R_2}, given by (2.2.2) gives sharpness for (2.3.10) in the disc U and for (2.3.11) in the disc $|z| \leq 2/(1 + \sqrt{5-4a})$, $a = \sqrt{1-1/R_2}$.

Remarks. (i) For $f \in CVG(R_1,R_2)$, $R_2 < \infty$, the upper bound on $|(f(z_1) - f(z_2))/(z_1-z_2)|$ in (2.3.11) is better than that in (2.3.7) for z_1,z_2 in the disc $|z| \le 2/(1 + \sqrt{5-4a})$ as well as in the annulus $2/(1 + \sqrt{5-4a}) < |z| < 1$, with $|z_1| < |z_2|$ where $a = \sqrt{1-1/R_2}$ and $|z|$ either increases or decreases on the line segment joining z_1,z_2.

(ii) For $f \in CVG(R_1,R_2)$ the upper bound on $|(f(z_1) - f(z_2))/(z_1 - z_2)|$ in (2.3.11) is better than the corresponding bound in (1.2.15) for z_1,z_2 in the disc $|z| \le 2/(1 + \sqrt{5-4a})$, $a = \sqrt{1-1/R_2}$, with $|z_1| < |z_2|$ and such that $|z|$ either increases or decreases on the line segment joining z_1 and z_2. For $z_1,z_2 \in U$, $|z_1| < |z_2|$ the lower bound in (2.3.10) is better than that in (1.2.15).

(iii) By choosing $z_1 = 0$, the inequalities (2.3.10), (2.3.11) give lower growth bound in U and upper growth bound in the disc $|z| \le 2/(1 + \sqrt{5-4a})$, $a = \sqrt{1-1/R_2}$ for $f \in CVG(R_1,R_2)$, $R_2 < \infty$. These growth bounds were found earlier by Ma et al. [71] for $CV(R_1,R_2)$ (cf. (1.2.42)) and continue to hold for $CVG(R_1,R_2)$ in view of Lemma 2.4.1.

2.4 In this section distortion and rotation theorems for the class $CVG(R_1,R_2)$ are found. We also prove in this section that $CVG(R_1,R_2)$, $R_2 < \infty$ is contained in $CV(\gamma)$ for $\gamma = 2R_2-1-2\sqrt{R_2^2-R_2}$ and the number γ is the largest possible such a number.

An analogue of the inequality (2.1.7) in terms of

$d^* = \sup_{\zeta \in \partial f(U)} |\zeta|$ for functions f in the class $CV(R_1, R_2)$ or $CVG(R_1, R_2)$ is not known. The following is a result in this direction:

<u>Theorem</u> 2.4.1. <u>If</u> $f \in CVG(R_1, R_2)$ <u>with</u> $0 < R_1 \le R_2 < \infty$, <u>then</u>

$$\frac{R_1 d^* |2R_1 - d^*|}{(R_1(1-r)+rd^*)^2} \le |f'(z)| \le \frac{R_2 d^*(2R_2 - d^*)}{(R_2 - |R_2 - d^*| r)^2} \tag{2.4.1}$$

<u>in the disc</u> $|z| = r \le 3 - \sqrt{8} \cong 0.171$. <u>The inequalities are sharp for</u> $R_1 = R_2$.

<u>Proof</u>. As in the proof of Theorem 2.3.1, we obtain

$$f(z) \ll \frac{Bz}{1-Az}$$

where, $B = d^*(2R_2 - d^*)/R_2$ and $A = (R_2 - d^*)/R_2$. This and the inequality (1.3.3) together give

$$|f'(z)| \le \frac{B}{(1 - |A| r)^2}$$

in the disc $|z| = r \le 3 - \sqrt{8}$. By substituting the values of A and B in this inequality, the right hand side inequality of (2.4.1) is obtained.

To prove the left hand side inequality of (2.4.1), we have, as in the proof of Theorem 2.3.1,

$$\frac{B^* z}{1-A^* z} \ll f(z)$$

where, $B^* = d^*(2R_1 - d^*)/R_1$ and $A = (R_1 - d^*)/R_1$. Therefore, by the inequality (1.3.3),

$$|f'(z)| \ge \left|\frac{B^*}{(1-A^*z)^2}\right|$$

$$\ge \frac{R_1 d^* |2R_1 - d^*|}{(R_1(1-r) + rd^*)^2}$$

in the disc $|z| = r \le 3 - \sqrt{8}$, which is the left hand side of the inequality (2.4.1).

For the functin F_{R_2}, given by (2.2.2), $R_1 = R_2$ and $d^* = 1/(1-a)$ so that

$$\frac{R_2 d^*(2R_2 - d^*)}{(R_2 - |R_2 - d^*|r)^2} = \frac{1}{(1-ar)^2} = |F'_{R_2}(r)|$$

and

$$\frac{R_1 d^* |2R_1 - d^*|}{(R_1(1-r) + rd^*)^2} = \frac{1}{(1+ar)^2} = |F'_{R_2}(-r)|$$

where, $a = \sqrt{1-1/R_2}$ so that equality is attained in the inequality (2.4.1).

<u>Remarks</u> (i) For $f \in CVG(R_1, R_2)$, the upper bound of $|f'(z)|$ in the inequality (2.4.1) is better than that in Goodman's inequality (2.1.7). The sharp function given in the proof of Theorem 2.4.1 is independent of the point under consideration

whereas the function used for the inequality (2.1.7) is dependent on the point.

(ii) From the proof of Theorem 2.4.1, it can be seen that the inequality (2.4.1) continues to remain true with d^* replaced by d everywhere, i.e., for $f \in CVG(R_1, R_2)$ with $0 < R_1 \le R_2 < \infty$, we have that

$$(2.4.2) \qquad \frac{R_1 d|2R_1 - d|}{(R_1 + |R_1 - d|r)^2} \le |f'(z)| \le \frac{R_2 d(2R_2 - d)}{(R_2(1-r) + rd)^2}$$

in the disc $|z| = r \le 3 - \sqrt{8}$.

(iii) For $f \in CVG(R_1, R_2)$ with $d^* \le R_2 \le 1/(12\sqrt{2} - 16)$ and r such that $\sqrt{R_2^2 - R_2} / R_2 \le r \le 3 - \sqrt{8}$, the upper bound of $|f'(z)|$ in the inequality (2.4.1) is better than that in the inequality (2.4.2).

(iv) For $f \in CVG(R_2, R_2)$ with $R_2 < \infty$, the lower bounds of $|f'(z)|$ in the inequalities (2.4.1) and (2.4.2) are equal by Theorem 2.2.1. Similarly, the upper bounds of $|f'(z)|$ are also equal in this case.

(v) For $f \in CVG(R_1, R_2)$ with $R_1 \le d \le d^* \le 2R_1$, the lower bound of $|f'(z)|$ in the inequality (2.4.2) is better than that in the inequality (2.4.1).

(vi) For $R_1 \ge 1$ and z in the disc $|z| \le 3 - \sqrt{8}$, the lower bound in (2.4.1) is smaller than that in (1.2.44) found by Ma et al. [71].

(vii) For $R_1 = R_2$, the lower bound in (2.4.1) and that in (1.2.44) found by Ma et al. [71] are the same. Similarly, the upper bound in (2.4.1) and that in (1.2.44) found by Ma et al. [71] are the same.

(viii) For $z = 0$, the upper bound in (2.4.2) and that in (1.2.46) found earlier by Mejia and Minda [78] are same. For $R_1 = R_2$ and $z = 0$, the upper bounds in (2.4.1) and (1.2.46) are same.

<u>Theorem</u> 2.4.2. <u>If</u> $f \in CVG(R_1, R_2)$, $0 < R_1 \le R_2 < \infty$, <u>then</u>

$$\frac{R_1 d^*|2R_1 - d^*|}{(R_1-(R_1-d^*)|z_2|)(R_1-(R_1-d^*)|z_1|)} \le \left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right| \tag{2.4.3}$$

$$\le \frac{R_2 d^*(2R_2-d^*)}{(R_2-|R_2-d^*|\ |z_2|)\ (R_2-|R_2-d^*|\ |z_1|)}$$

<u>for</u> z_1, z_2 <u>in the disc</u> $|z| \le 3 - \sqrt{8}$, $|z_1| < |z_2|$, <u>where the upper bound holds for</u> z_1, z_2 <u>such that</u> $|z|$ <u>either increases or decreases on the line segment joining</u> z_1 <u>and</u> z_2. <u>The inequality is sharp.</u>

<u>Proof</u>. The proof of the inequality (2.4.3) is similar to that of (2.3.10), (2.3.11) except that the distortion bounds in (2.4.1) in place of (1.2.44), (1.2.45) have to be used. The function F_{R_2}, defined by (2.2.2), gives sharpness.

<u>Remarks</u>. (i) For the sharp function F_{R_2}, given by (2.2.2), the corresponding bounds in (2.3.10), (2.3.11) and (2.4.3) become

identical in the disc $|z| \leq 3 - \sqrt{8}$.

(ii) It is observed in view of (2.4.2) that the statement of Theorem 2.4.2 with d^* replaced by d throughout continues to hold. Thus, if $f \in CVG(R_1,R_2)$, $0 < R_1 \leq R_2 < \infty$, then for distinct z_1,z_2 in the disc $|z| \leq 3 - \sqrt{8}$, $|z_1| < |z_2|$,

$$(2.4.4) \qquad \frac{R_1 d|2R_1-d|}{(R_1+|R_1-d|\ |z_2|)(R_1+|R_1-d|\ |z_1|)} \leq \left|\frac{f(z_1) - f(z_2)}{z_1-z_2}\right|$$

$$\leq \frac{R_2 d(2R_2-d)}{(R_2-(R_2-d)|z_2|)(R_2-(R_2-d)|z_1|)}$$

where $|z|$ either increases or decreases on the line segment joining z_1 and z_2.

(iii) For $d^* \leq R_2 \leq 1/(12\sqrt{2} - 16)$ and r such that $\sqrt{R_2^2 - R_2}/R_2 \leq r \leq 3 - \sqrt{8}$, the upper bound of $\left|\frac{f(z_1) - f(z_2)}{z_1-z_2}\right|$ in (2.4.3) is better than that in (2.4.4) for z_1,z_2 described there.

For $R_1 = R_2$, the upper bounds in (2.3.11), (2.4.3), (2.4.4) are the same for z_1,z_2 in the disc $|z| \leq 3 - \sqrt{8}$. Similarly, the lower bounds in (2.3.10),(2.4.3), (2.4.4) are the same for z_1,z_2 in $|z| \leq 3 - \sqrt{8}$.

For $R_1 \leq d \leq d^* \leq 2R_1$, the lower bound in (2.4.4) is better than that in (2.4.3).

Next, a rotation theorem for the class $CVG(R_1,R_2)$ is proved. The following observation is due to Styer and Wright [150].

<u>Lemma</u> 2.4.1. For $1 \le R_2 \le \infty$, the classes $CVG(0,R_2)$ and $CV(0,R_2)$ are identical.

<u>Proof</u>. It is proved in [150] that $CV(0,R_2) \subseteq CVG(0,R_2)$. Conversely, let $f \in CVG(0,R_2)$. Consider $1 \le R_2 < \infty$. Let a,b belong to $f(U)$. The domain $f(U)$ being convex (cf. Section 1.2), there exist $u,v \in f(U)$ such that u,v,a,b are distinct and that the intervals $[a,b] \subseteq [u,v] \subseteq f(U)$. By a necessary condition for $f \in CVG(0,R_2)$, (cf. Section 1.2, [150]) we obtain that there exist $a_1,b_1 \in f(U)$ such that a_1,b_1 are on $\partial\Delta_1 \cap \partial\Delta_2$, the points $u,v \in E(a_1,b_1;R_2) \equiv \Delta_1 \cap \Delta_2 \subseteq f(U)$ where Δ_1, Δ_2 are open discs of radius R_2. Thus, $a,b \in \Delta_i$, $i = 1,2$ and Δ_i being $1/R_2$-convex, by definition it follows that $E(a,b;R_2) \subseteq \Delta_i$, $i = 1,2$. Hence $E(a,b;R_2) \subseteq f(U)$. Now the relation (1.2.56) gives that f is in $CV(0,R_2)$. The case $R_2 = \infty$ is easily handled (cf. Section 1.2, [150]). Thus, the lemma is proved.

<u>Theorem</u> 2.4.3. <u>If</u> $f \in CVG(R_1,R_2)$ <u>with</u> $R_2 < \infty$, <u>then</u>

$$|\arg f'(z)| \le \int_0^r D(R_1,R_2,\rho,d^*)\, d\rho \tag{2.4.5}$$

<u>in the disc</u> $|z| = r \le 3 - \sqrt{8}$, <u>where</u>

$$D(c,b,\rho,t) = \frac{2\sqrt{(c(1-\rho)+\rho t)^2 b - (1-\rho^2)ct|2c-t|}}{(1-\rho^2)\sqrt{b}\,[c(1-\rho)+\rho t]} . \tag{2.4.6}$$

<u>Proof</u>. Let $f \in CVG(R_1, R_2)$. Using Lemma 2.4.1, we have $f \in CVG(R_1, R_2) \subseteq CVG(0, R_2) = CV(0, R_2)$. Further

$$f \in CV(0, R_2) \subseteq K(1/R_2, 1).$$

By the inequality (1.2.57), for $a \in U$, we have

$$(2.4.7) \qquad \left|\frac{f''(a)(1-|a|^2)}{2f'(a)} - \bar{a}\right| \le \sqrt{1 - \frac{(1-|a|^2)|f'(a)|}{R_2}} .$$

Multiplying (2.4.7) by $2|a|/(1-|a|^2)$, and using the lower distortion bound in (2.4.1) we obtain,

$$\left|\frac{af''(a)}{f'(a)} - \frac{2|a|^2}{1-|a|^2}\right| \le \frac{2|a|}{1-|a|^2}\sqrt{1 - \frac{(1-|a|^2)}{R_2} \cdot \frac{R_1 d^*|2R_1-d^*|}{(R_1(1-|a|)+|a|d^*)^2}}$$

for a in the disc $|z| \le 3 - \sqrt{8}$. Replacing $|a|$ by ρ in the above inequality, we get

$$-\rho D(R_1, R_2, \rho, d^*) \le \operatorname{Im}\left(\frac{af''(a)}{f'(a)} - \frac{2\rho^2}{1-\rho^2}\right) \le \rho D(R_1, R_2, \rho, d^*)$$

where $D(c, b, \rho, t)$ is as in (2.4.6). Thus

$$(2.4.8) \qquad -D(R_1, R_2, \rho, d^*) \le \frac{\partial}{\partial\rho} \arg f'(a) \le D(R_1, R_2, \rho, d^*),$$

since,

$$\operatorname{Im}\left(\frac{af''(a)}{f'(a)} - \frac{2\rho^2}{1-\rho^2}\right) = \rho \frac{\partial}{\partial\rho} \arg f'(a).$$

Now, integrating the terms in the inequality (2.4.8) along the straight line path $a = 0$ to $a = re^{i\theta}$, the required inequality

(2.4.5) follows.

Remarks. (i) Using the lower distortion bound in (1.2.44) in place of (2.4.1), the proof of Theorem 2.4.3 gives that for f in $CVG(R_1,R_2)$, $R_2 < \infty$, and $z \in U$,

$$(2.4.9) \qquad |\arg f'(z)| \le \ln \frac{(1+Ar)^2}{1-r^2}$$

where, $|z| = r$ and $A = \sqrt{1-1/R_2}$. This was obtained earlier by Wirths in [158].

For z in a neighbourhood of the origin, the bound in (2.4.5) is better than that in (2.4.9) when $f \in CVG(R_1,R_2)$, $R_2 > 2R_1 \ge d^*$ and $R_2 > 4/3$.

(ii) Using the lower bound in (2.4.2) in place of (2.4.1) in the proof of Theorem 2.4.3, we obtain that if f is in $CVG(R_1,R_2)$, $R_2 < \infty$, then for z in the disc $|z| \le 3 - \sqrt{8}$,

$$(2.4.10) \qquad |\arg f'(z)| \le \int_0^r E(R_1,R_2,\rho,d)\, d\rho$$

where $E(a,b,\rho,t) = \dfrac{2\sqrt{(a+|a-t|\rho)^2 b - (1-\rho^2)at\,|2a-t|}}{(1-\rho^2)\sqrt{b}\,(a+|a-t|\rho)^2}$.

The bound in (2.4.10) is better than that in (2.4.5) when $f \in CVG(R_1,R_2)$, with $R_1 \le d \le d^* \le 2R_1$.

For $f \in CVG(R_2,R_2)$, the bounds in (2.4.5), (2.4.9) and (2.4.10) are equal.

The following theorem gives bounds on the curvature $k(f;z)$, when $f \in CVG(R_1,R_2), R_2 < \infty$.

<u>Theorem</u> 2.4.4. <u>If</u> $f \in CVG(R_1,R_2)$, $1 \le R_2 < \infty$, <u>then, for</u> $|z| = r$,

$$\frac{(1-ar)^2}{1-r^2}\left(\frac{1+r^2}{r} - 2\,\frac{a+r}{1+ar}\right) \le k(f;z) \le \frac{(1+ar)^2}{1-r^2}\left(\frac{1+r^2}{r} + 2\,\frac{a+r}{1+ar}\right),$$

<u>where, the upper inequality holds for</u> $z \in U\backslash\{0\}$ <u>and the lower inequality holds in the annulus</u> $0 < |z| \le 2/(1 + \sqrt{5-4a})$. <u>Further, for</u> z <u>in the annulus</u> $|z| > 2/(1 + \sqrt{5-4a})$,

$$k(f;z) \ge (1-a)\ (2+(1-a)r)\ \left(1 + r^2 - 2r\,\frac{a+r}{1+ar}\right)$$

where, $a = \sqrt{1-1/R_2}$.

<u>Proof</u>. From (2.4.7), we have for $z \in U$,

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}\sqrt{1 - \frac{(1-r^2)|f'(z)|}{R_2}}\ , \quad |z|=r.$$

By the distortion bounds (1.2.44) for $CVG(0,R_2)$, we have

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}\sqrt{1 - \frac{(1-r^2)}{R_2(1+ar)^2}}$$

where, $a = \sqrt{1-1/R_2}$. Thus,

$$\frac{1}{1-r^2}\left(1+r^2 - 2r\left(\frac{a+r}{1+ar}\right)\right) \le \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \tag{2.4.11}$$

$$\le \frac{1}{1-r^2}\left(1+r^2 + 2r\left(\frac{a+r}{1+ar}\right)\right).$$

Now the assertion of the theorem follows by combining (2.4.11) and the distortion bounds in (1.2.44), (1.2.45) for $CVG(0,R_2)$.

Finally, for functions f in $CVG(R_1,R_2)$, $R_2 < \infty$, a number γ is obtained such that f is convex of order γ.

Theorem 2.4.5. *If* $f \in CVG(R_1,R_2)$, $R_2 < \infty$, *then* $f \in CV((1-A)/(1+A))$ *where* $A = \sqrt{1-1/R_2}$. *The result is sharp.*

Proof. Let $f \in CVG(R_1,R_2)$, $R_2 < \infty$. Without loss of generality [44] we may assume that $f(z)$ is analytic in $\bar{U}$. Since f is in $CVG(R_1,R_2) \subseteq CVG(0,R_2) = CV(0,R_2)$, by Lemma 2.4.1 and $CV(0,R_2) \subseteq C(1/R_2)$ it follows from (1.2.54) that for $z \in U\backslash\{0\}$,

$$k(f;z) > \frac{1}{R_2} ,$$

equivalently,

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{|zf'(z)|}{R_2} .$$

Thus, the lower distortion bound in (1.2.44) for $f \in CV(0,R_2)$ gives that for $z \in U\backslash\{0\}$,

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{|z|}{R_2(1+A|z|)^2} \geq \frac{|z|}{R_2(1+A)^2} .$$

Hence, on $|z| = 1$,

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \frac{1}{R_2(1+A)^2}$$

$$= \frac{1-A}{1+A}$$

where, $A = \sqrt{1-1/R_2}$. Thus, it follows that $f \in CV((1-A)/(1+A))$. The function $F_{R_2} \in CVG(R_1,R_2)$, $1 \le R_2 < \infty$, $0 \le R_1 \le R_2 < \infty$, where F_{R_2} is given by (2.2.2), shows that the number $(1-A)/(1+A)$, can not be replaced by a larger number in the result where $A = \sqrt{1-1/R_2}$.

<u>Remark</u>. In [44], Goodman conjectured that if $f \in CV(R_1,R_2)$, $R_2 < \infty$, then $f \in CV(\gamma)$ where $\gamma = 2R_2 - 1 - 2\sqrt{R_2^2 - R_2}$ and γ is the largest possible such a number. Theorem 2.4.5 solves an analogous problem for $CVG(R_1,R_2)$, $R_2 < \infty$ with the same $\gamma = (1-A)/(1+A)$, where $A = \sqrt{1-1/R_2}$.

CHAPTER III

CONVEX FUNCTIONS OF BOUNDED α-TYPE

3.1 In this chapter we introduce the notion of α-curvature for functions analytic and locally univalent in the unit disc U and find distortion bounds, bounds on the second and third Taylor series coefficients, a sharp γ such that $\text{Re}\,(1 + zf''(z)/f'(z)) > \gamma$ in U etc., for functions in the resulting classes $CV_\alpha(R_1,R_2)$ and $C_\alpha(K)$.

We have the following definitions :

Definition 3.1.1. Let $f \in A$ be locally univalent in U and $\alpha < 1$. The α-curvature, $k_\alpha(z)$ of $f(|z| = r)$ at the point $f(z)$ is defined as

$$k_\alpha(z) = \frac{\text{Re}(1 + zf''(z)/(1-\alpha)f'(z))}{|z|\ |f'(z)|^{1/(1-\alpha)}} \tag{3.1.1}$$

where, $z = re^{i\theta}$ and $0 < r < 1$.

It follows that

$$k_\alpha(z) = \frac{|z|^{\alpha/(1-\alpha)}((d\tau/dt) - \alpha)}{(1-\alpha)(ds/dt)^{1/(1-\alpha)}}$$

where s is the arc length on $f(|z| = r)$ and τ is the argument of

the tangent to $f(|z| = r)$ at the point $f(z)$.

We call the reciprocal $\rho_\alpha(z) = 1/k_\alpha(z)$, of $k_\alpha(z)$ to be the radius of α-curvature of $f(|z| = r)$ at the point $f(z)$, $\alpha < 1$. When stress is needed on the function, we denote $k_\alpha(z)$ and $\rho_\alpha(z)$ by $k_\alpha(f;z)$ and $\rho_\alpha(f;z)$ respectively. The 0-curvature and radius of 0-curvature of $f(|z| = r)$ at the point $f(z)$ coincide with the usual notion of curvature $k(z)$ and radius of curvature $\rho(z)$.

The function

(3.1.2) $$l_\alpha(z) = \begin{cases} \dfrac{1-(1-az)^{2\alpha-1}}{a(2\alpha-1)}, & \alpha \neq 1/2 \\[2ex] \dfrac{1}{a} \log \dfrac{1}{1-az}, & \alpha = 1/2 \end{cases}$$

$0 < a < 1$ has the α-curvature $k_\alpha(z) \equiv \dfrac{1-a^2|z|^2}{|z|}$ for every α in $0 \leq \alpha < 1$.

The function

(3.1.3) $$p_a(z) = z - az^2$$

$0 \leq a \leq 1/2$, has the α-curvature

$$k_\alpha(z) = \frac{1-\alpha-2(3-2\alpha)ar\cos\theta + 4(2-\alpha)a^2r^2}{(1-\alpha)r(1-4ar\cos\theta+4a^2r^2)^{(3-2\alpha)/(2(1-\alpha))}}$$

for $z = re^{i\theta}$ and $\alpha < 1$.

Now, we introduce a subclass $CV_\alpha(R_1,R_2)$ of S as follows:

For $f \in S$, let

(3.1.4) $\rho_*(\alpha,r) = \min_{|z|=r} \rho_\alpha(f;z)$; $\rho^*(\alpha,r) = \max_{|z|=r} \rho_\alpha(f;z)$,

(3.1.5) $R_*(\alpha) = \liminf_{r\to 1^-} \rho_*(\alpha,r)$; $R^*(\alpha) = \limsup_{r\to 1^-} \rho^*(\alpha,r)$

for $0 \le \alpha < 1$.

The function $l_\alpha(z)$ in (3.1.2) has $\rho_*(\alpha,r)=\rho^*(\alpha,r)=r/(1-a^2r^2)$ and $R_*(\alpha) = R^*(\alpha) = 1/(1-a^2)$.

The function $p_a(z)$ in (3.1.3) has

$$\rho_*(\alpha,r) = r(1-4a^2r^2)^{1/2(1-\alpha)},$$

$$\rho^*(\alpha,r) = \max\ (1/k_\alpha(r),\ \ 1/k_\alpha(-r)),$$

$$R_*(\alpha) = (1-4a^2)^{1/(2(1-\alpha))},$$

$$R^*(\alpha) = (1-\alpha)(1-2a)^{(3-2\alpha)/(1-\alpha)}/(1-\alpha-2(3-2\alpha)a + 4(2-\alpha)a^2),$$

where $0 \le a \le (1-\alpha)/\ (2(2-\alpha))$, $0 \le \alpha < 1$.

<u>Definition</u> 3.1.2. A function $f \in S$ is said to be in the class $CV_\alpha(R_1,R_2)$, $0 \le \alpha<1$, if $R_1 \equiv R_1(\alpha) \le R_*(\alpha)$ and $R^*(\alpha) \le R_2(\alpha) \equiv R_2$ where, $R_*(\alpha)$, $R^*(\alpha)$ are as in (3.1.5) and R_1,R_2 are fixed in $[0,\infty]$ such that $R_1 \le R_2$.

We have $CV_o(R_1,R_2) = CV(R_1,R_2)$ (cf. Section 1.2). The class of functions for which $R_1 = R_*(\alpha)$ and $R_2 = R^*(\alpha)$ is denoted by $\overline{CV}_\alpha(R_1,R_2)$. For example, the function $l_\alpha(z)$ in (3.1.2) is in the

class $\overline{CV}_\alpha(1/(1-a^2),\ 1/(1-a^2))$ when $0 < a < 1$. The function $p_a(z) = z - az^2$, $0 \le a \le (1-\alpha)/(2(2-\alpha))$ is in the class $\overline{CV}_\alpha(\beta_1,\beta_2)$, where $\beta_1 \equiv \beta_1(a,\alpha) = (1-4a^2)^{1/2(1-\alpha)}$ and

$$\beta_2 \equiv \beta_2(a,\alpha) = \frac{(1-\alpha)(1-2a)^{(3-2\alpha)/(1-\alpha)}}{1-\alpha-2(3-2\alpha)a+4(2-\alpha)a^2}.$$

For $R_1^* \equiv R_1^*(\alpha) \le R_1$ and $R_2 \le R_2^*(\alpha) \equiv R_2^*$, we note that the class $CV_\alpha(R_1,R_2)$ is contained in the class $CV_\alpha(R_1^*,R_2^*)$. The function $l_\alpha(z)$ in (3.1.2) shows that, for $R_2 > 1$,

$$CV_\alpha(R_1,R_2) \subsetneq CV_\alpha(R_1^*,R_2) \qquad \text{if} \quad R_1^* < R_1 \le R_2,$$

$$CV_\alpha(R_1,R_2) \subsetneq CV_\alpha(R_1,R_2^*) \qquad \text{if} \quad R_1 \le R_2 < R_2^*.$$

We have $\displaystyle\bigcup_{0\le R_1\le R_2\le\infty} CV_\alpha(R_1,R_2) = CV(\alpha)$ (cf. Section 1.2). Also, $CV_\alpha(R_1,R_2) \subseteq CV(o,\infty)$. For $R_2 < \infty$, $CV_\alpha(\infty,\infty)$ is not contained in $CV(R_1,R_2)$ since the former class contains unbounded function $l_\alpha(z)$ given by (3.1.2) with $a = 1$, and the functions in the latter class are bounded.

<u>Definition</u> 3.1.3. A function $f \in CV_\alpha(R_1,R_2)$ with $0 < R_1 \le R_2 < \infty$, $0 < \alpha < 1$, is called a <u>convex function of bounded α-type</u>.

It will be seen in Section 3.2 that $f \in CV_\alpha(R_1,R_2)$, if and only if, $g(z) \equiv \int_0^z (f'(\tau))^{1/(1-\alpha)}\, d\tau \in CV(R_1,R_2)$, for $0 \le \alpha < 1$, $R_2 < \infty$.

We also investigate the class $C_\alpha(K)$, defined below, that reduces to the Wirths' class $C(K)$ (cf. Section 1.2), if $\alpha = 0$.

<u>Definition</u> 3.1.4. Let $K > 0$ and f be analytic and locally univalent in U. Then, f is said to be in the class $C_\alpha(K)$, $\alpha < 1$, if

$$\liminf_{|z|\to 1} k_\alpha(f;z) \geq K. \tag{3.1.6}$$

It is easily seen that $CV_\alpha(R_1,R_2) \subseteq C_\alpha(1/R_2)$ when $0 < R_2 < \infty$. For $K > K^* > 0$ we have $C_\alpha(K) \subseteq C_\alpha(K^*)$. The function $l_\alpha(z)$ in (3.1.2) shows that $C_\alpha(K) \subsetneqq C_\alpha(K^*)$ for $0 < K^* < K < 1$.

A result found in Section 3.3 will show that $f \in C_\alpha(K)$, $K > 0$, $\alpha < 1$, if and only if, $g(z) \equiv \int_0^z (f'(\tau))^{1/(1-\alpha)} d\tau \in C(K)$. Wirths [160] found that if $f \in C(K)$, $K > 0$, is analytic, locally univalent in a disc $|z| < 1 + \epsilon$, $\epsilon > 0$ and the function $t(z)$ mapping $\{z \in \mathbb{C} : |z| = 1\}$ into $\mathbb{R}$ is continuous and nonnegative, then, for any $a \in U$,

$$\left|\frac{1}{2\pi} \int_0^{2\pi} Kf'(e^{i\theta})(1-\bar{a}e^{i\theta})^2\, t(e^{i\theta})d\theta\right| \tag{3.1.7}$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(1 + \frac{e^{i\theta} f''(e^{i\theta})}{f'(e^{i\theta})}\right)|1-\bar{a}e^{i\theta}|^2\, t(e^{i\theta})d\theta$$

and equality holds in (3.1.7), only for the function

$$f(z) = \frac{e^{i\phi}}{K}\left(\frac{z+a}{1+\bar{a}z}\right) + b, \qquad \phi \in \mathbb{R},\ |a| < 1,\ b \in \mathbb{C}. \tag{3.1.8}$$

Using (3.1.7), with different special choices of function $t(z)$, Wirths [160] derived several old and new results for $CV(R_1,R_2)$. Thus, he proved that if $f \in C(K)$, $K > 0$, then for $z \in U$, the inequality (1.2.43) continues to hold with R_2 replaced by $1/K$ and equality in (1.2.43) holds for f given by (3.1.8). Moreover, if $f(z) = \sum_{k=0}^{\infty} c_k z^k \in C(K)$, $K > 0$, then

$$|c_2|^2 \le |c_1|^2 (1-K|c_1|). \tag{3.1.9}$$

Equality holds in (3.1.9) only for $f(z)$ given by (3.1.8). We observe that the inequality (3.1.9) is the same as the inequality (1.2.39) if $K = 1/R_2 > 0$, $c_1 = 1$ and $k = 2$.

For the class $C_\alpha(K)$, some of the results found in the present chapter give the analogues of Wirths' inequalities (3.1.7) and (3.1.9).

The following maximum-minimum principle [159] for locally univalent functions is quite helpful in the study of class $C(K)$:

Let $f(z)$ be analytic at $z_0 \in U$.

(3.1.10) If $f'(z_0) \ne 0$, the function

$$u_f(z) = \frac{1 - \left|\frac{f''(z)}{2f'(z)}(1-|z|^2) - \bar{z}\right|^2}{|f'(z)|\,(1-|z|^2)}$$

defined in a neighbourhood of z_0 has a local minimum at z_0, if and only if, f is a Mobius transformation.

$$(3.1.11)\quad \begin{cases} \text{If } f'(z_0) \neq 0 \text{ and } f''(z_0)(1-|z_0|^2) \neq 2\bar{z}_0 f'(z_0),\ u_f \text{ has a} \\ \text{local maximum at } z_0, \text{ if and only if, } f(z) \text{ is a Mobius} \\ \text{transformation.} \end{cases}$$

Using the above maximum-minimum principle for locally univalent functions, Wirths [159] gave an alternate proof of the inequality (3.1.9).

In the present chapter, for the class $C_\alpha(K)$, we first find an analogue of the above Wirths' maximum-minimum principle and then, using this analogue, find (i) a sharp bound on the modulus of the second Taylor coefficient for functions in $C_\alpha(K)$ and (ii) a rotation theorem for the class $CV_\alpha(R_1,R_2)$.

The Schwarzian derivative $[f]_z$ of an analytic, locally univalent function f in U is defined as (p.258, [26]) :

$$[f]_z(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

An upper bound of $|[f]_z(z)|$ is known [159], if $f \in C(K)$, $K > 0$. Thus, for $f \in C(K)$ and $z \in U$,

$$(3.1.12)\quad \frac{1}{2}|[f]_z(z)|(1-|z|^2)^2(1 - \frac{K}{2}|f'(z)|(1-|z|^2))$$
$$\leq 1 - \left|\frac{f''(z)}{2f'(z)}(1-|z|^2) - \bar{z}\right|^2 - K|f'(z)|(1-|z|^2).$$

In inequality (3.1.12), equality for one $z \in U$ implies equality for all $z \in U$ and this can happen only for functions f(z) given by

(3.1.8). Thus, it follows from (3.1.12) [159] that if $f(z) = \sum_{k=0}^{\infty} c_k z^k \in C(K)$, $K > 0$, then

$$|c_3| \le |c_1| \, (1-K|c_1|). \tag{3.1.13}$$

Equality holds in (3.1.13) only for functions f(z) in (3.1.8). For the class $C_\alpha(K)$, some of our results found in the present chapter give the analogues of Wirths' inequalities (3.1.12) and (3.1.13).

A necessary and sufficient condition for C(K), K > 0 is determined in [160]. Thus, $f(z) = \sum_{k=0}^{\infty} c_k z^k \in C(K)$, if and only if,

$$f\left(\frac{z+a}{1+\bar{a}z}\right) \in C(K), \ a \in U, \text{ and the inequality (3.1.9) holds.} \tag{3.1.14}$$

It will be seen in the sequel that an analogue of (3.1.14) holds for the class $C_\alpha(K)$.

In Section 3.2 an integral operator that transforms functions in the class $CV_\alpha(R_1,R_2)$ into functions of the class $CV(R_1,R_2)$ (cf. Section 1.2) is studied. A similar integral operator between the classes $ST(R_1,R_2)$ (cf. Section 1.2) and $CV_\alpha(R_1,R_2)$ is also studied in this section. The former integral operator is helpful in (i) obtaining a sharp γ such that a function in $CV_\alpha(R_1,R_2)$ belongs to $CV(\gamma)$ and (ii) finding distortion bounds for the class $CV_\alpha(R_1,R_2)$. A growth theorem for the class $CV_\alpha(R_1,R_2)$ and bounds on the functional

$|(f(z_1) - f(z_2))/(z_1-z_2)|$ for $f \in CV_\alpha(R_1,R_2)$, are also obtained in the same section. Finally, in Section 3.3, the bounds on the second and third Taylor series coefficients for functions in the class $C_\alpha(K)$, a distortion theorem and necessary and sufficient conditions for a function to be in the class $C_\alpha(K)$ are found. A rotation theorem is also derived for $f \in CV_\alpha(R_1,R_2)$ in this section.

3.2 In this section an integral operator which transforms functions in $CV_\alpha(R_1,R_2)$ into functions in $CV(R_1,R_2)$ (cf. Section 1.2) is investigated. Using this transform, a sharp γ such that $CV_\alpha(R_1,R_2)$ is contained in $CV(\gamma)$ is obtained. With the help of the same transform, distortion bounds are determined for the class $CV_\alpha(R_1,R_2)$. A similar integral operator between the classes $CV_\alpha(R_1,R_2)$ and $ST(R_1,R_2)$ (cf. Section 1.2) is also studied. Finally, in this section a growth theorem for the class $CV_\alpha(R_1,R_2)$ and bounds on the functional $|(f(z_1) - f(z_2))/(z_1-z_2)|$ are determined for $f \in CV_\alpha(R_1,R_2)$ and certain distinct z_1,z_2 in U.

In the following result it is observed that the classes $CV(R_1,R_2)$ and $CV_\alpha(R_1,R_2)$ are related through an integral operator.

<u>Theorem</u> 3.2.1 <u>Let a function</u> $f \in A_1$. <u>Then, the function</u> $g(z) \equiv \int_0^z (f'(\tau))^{1/(1-\alpha)}\, d\tau \in CV(R_1,R_2)$, <u>if and only if</u>, $f \in CV_\alpha(R_1,R_2)$ <u>where</u> $0 \le \alpha < 1$, <u>and</u> $R_2 < \infty$.

Proof. For $g(z) \equiv \int_0^z (f'(\tau))^{1/(1-\alpha)}\, d\tau$, we have

$$1+\frac{zg''(z)}{g'(z)} = 1+\frac{zf''(z)}{(1-\alpha)f'(z)} .$$

Hence,

(3.2.1) $\qquad g \in CV$, if and only if, $f \in CV(\alpha)$.

Moreover, for $z \in U\setminus\{0\}$,

(3.2.2) $\qquad k(g;z) = k_\alpha(f;z)$.

Now the definitions of the classes $CV(R_1,R_2)$ and $CV_\alpha(R_1,R_2)$, relation (3.2.1) and equation (3.2.2) together give the theorem.

Remark. For the class $CV_\alpha(R_1,R_2)$, R_2 is always greater than or equal to 1, since it follows from Theorem 3.2.1 that if $f \in CV_\alpha(R_1,R_2)$, $R_2 < \infty$, $0 < \alpha < 1$, then the function g defined in the theorem is in the class $CV(R_1,R_2)$ and for the class $CV(R_1,R_2)$, we have $R_2 \geq 1$ (cf. (1.2.37)).

Corollary 3.2.1. If $f \in CV_\alpha(R_1,R_2)$ with $0 < \alpha < 1$, then

(3.2.3) $$|f'(re^{i\theta})| \leq \left(\frac{R_2}{1-r^2}\right)^{1-\alpha}$$

where $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$. The inequality is sharp.

Proof. Inequality (2.1.7) and Theorem 3.2.1 together give the required inequality. The function l_α defined in (3.1.2), with $a = \sqrt{1-1/R_2}$, $R_2 \geq 1$, is in the class $CV_\alpha(R_2,R_2) \subseteq CV_\alpha(R_1,R_2)$. For this function, $l'_\alpha(z) = (1-az)^{2\alpha-2}$ so that

$$|l'_\alpha(a)| = (1-a^2)^{2(\alpha-1)} = (R_2/(1-r^2))^{1-\alpha}.$$

Thus, sharpness in the inequality (3.2.3) is attained at the point $re^{i\theta} = a$.

Corollary 3.2.2. If $f \in CV_\alpha(R_1,R_2)$ with $R_2 < \infty$, and $0 \le \alpha < 1$ then

$$(3.2.4)\qquad \left[\frac{R_1 d^*|2R_1-d^*|}{(R_1(1-r)+rd^*)^2}\right]^{1-\alpha} \le |f'(re^{i\theta})| \le \left[\frac{R_2 d^*(2R_2-d^*)}{(R_2-|R_2-d^*|r)^2}\right]^{1-\alpha}$$

in $r \le 3 - \sqrt{8}$, $0 \le \theta \le 2\pi$ where, $d^* = d^*(g) = \max\limits_{\zeta\in\partial g(U)} |\zeta|$; $g(z) = \int_0^z (f'(\tau))^{1/(1-\alpha)}\, d\tau$. The inequality (3.2.4) is sharp.

Proof. Since the class $CV(R_1,R_2) \subseteq CVG(R_1,R_2)$, Theorems 2.4.1 and 3.2.1 together give the required inequality. The function $l_\alpha(z)$ given in (3.1.2), with $a = \sqrt{1-1/R_2}$, $R_2 \ge 1$, is in $CV_\alpha(R_2,R_2) \subseteq CV_\alpha(R_1,R_2)$. For this function,

$$g(z) = \int_0^z (l'_\alpha(z))^{1/(1-\alpha)}\, d\tau = z/(1-az); \qquad d^* = d^*(g) = 1/(1-a) > R_2$$

so that

$$R_1d^*|2R_1-d^*|/(R_1(1-r)+rd^*)^2 = 1/(1+ar)^2 = |l'_\alpha(-r)|^{1/(1-\alpha)}$$

and

$$R_2d^*(2R_2-d^*)/(R_2-|R_2-d^*|r)^2 = 1/(1-ar)^2 = |l'_\alpha(r)|^{1/(1-\alpha)}.$$

This proves the corollary.

Remark. We note that the sharpness function given in the proof of Corollary 3.2.1 is dependent on the point at which sharpness is attained in the inequality (3.2.3), whereas the sharpness function given in Proof of Corollary 3.2.2 is independent of any such point.

Corollary 3.2.3. If $f \in CV_\alpha(R_1,R_2)$ with $R_2 < \infty$ and $0 \le \alpha < 1$, then

$$(3.2.5)\qquad \left[\frac{R_1 d|2R_1-d|}{(R_1+|R_1-d|r)^2}\right]^{1-\alpha} \le |f'(re^{i\theta})| \le \left[\frac{R_2 d(2R_2-d)}{(R_2(1-r)+rd)^2}\right]^{1-\alpha}$$

in the disc $r \le 3-\sqrt{8}$; $0 \le \theta \le 2\pi$, where $d = d(g) = \min\limits_{\zeta\in\partial g(U)} |\zeta|$; $g(z) = \int_0^z (f'(\tau))^{1/(1-\alpha)}\, d\tau$. The inequality is sharp.

Proof. The class $CV(R_1,R_2) \subseteq CVG(R_1,R_2)$, inequality (2.4.2) and Theorem 3.2.1 together give the required inequality. For the function $l_\alpha(z)$ given by (3.1.2), $R_1 = R_2 = 1/(1-a^2)$ and $g(z) = z/(1-az)$. Hence, $d = d(g) = 1/(1+a)$ so that

$$R_1 d|2R_1-d|/(R_1+|R_1-d|r)^2 = 1/(1+ar)^2 = |l'_\alpha(-r)|^{1/(1-\alpha)}$$

and

$$R_2 d(2R_2-d)/(R_2(1-r)+rd)^2 = 1/(1-ar)^2 = |l'_\alpha(r)|^{1/(1-\alpha)}.$$

The proof of the corollary is therefore complete.

Remarks (i) For $f \in CV_\alpha(R_1,R_2)$, $R_2 < \infty$, $0 \le \alpha < 1$, with $d^* \le R_2 \le 1/(12\sqrt{2}-16)$ and r such that $\sqrt{R_2^2-R_2} \le r \le 3-\sqrt{8}$, the upper bound of $|f'(z)|$ in the inequality (3.2.4) is better than

that in the inequality (3.2.5), where d^* is as in Corollary 3.2.2.

(ii) For $f \in CV_\alpha(R_2, R_2)$ with $R_2 < \infty$, $0 \le \alpha < 1$, the lower bounds of $|f'(z)|$ in the inequalities (3.2.4) and (3.2.5) are equal by Theorems 2.2.1 and 3.2.1. Similarly, the upper bounds of $|f'(z)|$ are also equal in this case.

(iii) For $f \in CV_\alpha(R_1, R_2)$ with $R_1 \le d \le d^* \le 2R_1$, the lower bound of $|f'(z)|$ in the inequality (3.2.5) is better than that in the inequality (3.2.4) where d and d^* are as in corollaries 3.2.2 and 3.2.3.

<u>Corollary</u> 3.2.4. If $f \in CV_\alpha(R_1, R_2)$, $R_2 < \infty$, $0 < \alpha < 1$, then

$$(3.2.6) \qquad \frac{1}{(1+Ar)^{2(1-\alpha)}} \le |f'(z)| \le \frac{1}{(1-Ar)^{2(1-\alpha)}}$$

where, the left hand side inequality in (3.2.6) holds for $z \in U$ and the right hand side inequality in (3.2.6) holds in the disc $|z| \le \dfrac{2}{1+\sqrt{5-4A}}$ where $|z| = r$, $A = \sqrt{1-1/R_2}$.

Further for z in the annulus, $\dfrac{2}{1+\sqrt{5-4A}} < |z| < 1$,

$$(3.2.7) \qquad |f'(z)| \le \frac{1}{[(1-A)(1-r^2)r(2+(1-A)r)]^{1-\alpha}}\,.$$

<u>Proof</u>. The Corollary is a direct consequence of the combination of Theorem 3.2.1 and the distortion bounds (cf. (1.2.44), (1.2.45)) for $CV(R_1, R_2)$. The function $l_\alpha(z)$, defined by (3.1.2), gives sharpness in (3.2.6).

Remarks. (i) For $R_1 \geq 1$ and $|z| \leq 3 - \sqrt{8}$, the lower bound in (3.2.4) is smaller than that in (3.2.6).

(ii) For $R_1 = R_2$, the lower bound in (3.2.4) and that in (3.2.6) are the same. Similarly, the upper bound in (3.2.4) and that in (3.2.6) are the same.

Next, we show that functions in the class $CV_\alpha(R_1,R_2)$, $R_2 < \infty$ are convex of order $((1-\alpha)(1-A)/(1+A) + \alpha)$ where, $A = \sqrt{1-1/R_2}$.

Theorem 3.2.2. *If* $f \in CV_\alpha(R_1,R_2)$ *with* $R_2 = R_2(\alpha) < \infty$, *then*

$$CV_\alpha(R_1,R_2) \subseteq CV((1-\alpha)\frac{1-A}{1+A} + \alpha)$$

where, $A = \sqrt{1-1/R_2}$ *and* $0 < \alpha < 1$. *The result is sharp*.

Proof. Let $f \in CV_\alpha(R_1,R_2)$. By Theorem 3.2.1, the function

$$g(z) = \int_0^z (f'(\tau))^{1/(1-\alpha)}\, d\tau \in CV(R_1,R_2).$$

Since $g \in CV(R_1,R_2) \subseteq CVG(R_1,R_2)$, Theorem 2.4.5 gives that $g(z)$ is in $CV((1-A)/(1+A))$ where, $A = \sqrt{1-1/R_2}$. Thus, for $z \in U$,

$$\mathrm{Re}(1 + \frac{zg''(z)}{g'(z)}) = \mathrm{Re}(1 + \frac{zf''(z)}{(1-\alpha)f'(z)}) > \frac{1-A}{1+A}\ .$$

Hence,

$$\mathrm{Re}(1 + \frac{zf''(z)}{f'(z)}) > (1-\alpha)\frac{1-A}{1+A} + \alpha.$$

The function $l_\alpha(z)$ defined by (3.1.2) gives sharapness of the

result.

We observe that for $R_2 < \infty$, $0 < \alpha < 1$,

(3.2.8) $$CV(\alpha) \not\subseteq CV_{\alpha}(R_1,R_2)$$

by considering the following example. Suppose if possible that the function $l_{\alpha}(z) \in CV(\alpha)$ with $a = 1$, (cf. (3.1.2)) is also in $CV_{\alpha}(R_1,R_2)$. Then, by Theorem 3.2.1, we have that the function

$$g(z) = \int_0^z (l'_{\alpha}(\tau))^{1/(1-\alpha)}\, d\tau$$

$$= \frac{z}{1-z}$$

is in $CV(R_1,R_2)$, so that $R^* \le R_2 < \infty$ where, R^* is as in (1.2.33). But, $g(z) = z/(1-z)$ is in $\overline{CV}(\infty,\infty)$ so that $R^* = \infty$. This is a contradiction and thus (3.2.8) is established.

Next, we determine bounds on the functional $|(f(z_1) - f(z_2))/(z_1-z_2)|$ for f in the class $CV_{\alpha}(R_1,R_2)$ and certain distinct z_1,z_2 in U.

<u>Theorem</u> 3.2.3. <u>If</u> $f \in CV_{\alpha}(R_1,R_2)$, $R_2 < \infty$, $0 < \alpha < 1$, <u>then</u>

(3.2.9) $$\left|\frac{f(z_1) - f(z_2)}{z_1-z_2}\right| \ge \frac{J(A,\alpha,r_1,r_2)}{r_2-r_1}$$

<u>for</u> z_1,z_2 <u>in</u> U <u>where</u>, $J(A,\alpha,t,u) = I(A,\alpha,u) - I(A,\alpha,t)$,

$$J(A,1/2,t,u) = (1/A)\ln((1+Au)/(1+At)),$$

$I(A,\alpha,t) = \dfrac{1}{A(2\alpha-1)(1+At)^{1-2\alpha}}$, $\alpha \neq 1/2$, $A = \sqrt{1-1/R_2}$, $|z_i| = r_i$, $i = 1,2$ *and* $r_1 < r_2$.

Further, if $|z|$ *either increases or decreases on the line segment joining* z_1 *and* z_2, $|z_1| < |z_2|$, *then*

$$(3.2.10) \quad \left|\frac{f(z_1) - f(z_2)}{z_1-z_2}\right| \le \begin{cases} \dfrac{-J(A,\alpha,-r_1,-r_2)}{r_2-r_1}, & r_2 \le \dfrac{2}{1+\sqrt{5-4A}} \\[2ex] \dfrac{L(A,\alpha,r_1,r_2)}{r_2-r_1}, & \dfrac{2}{1+\sqrt{5-4A}} < r_1 < r_2 < 1 \end{cases}$$

where $L(A,\alpha,t,u) = \displaystyle\int_t^u \frac{d\rho}{[(1-A)(1-\rho^2)\rho(2+(1-A)\rho)]^{1-\alpha}}$. *The inequality* (3.2.9) *is sharp in* U *and* (3.2.10) *is sharp in* $r_2 \le 2/(1+\sqrt{5-4A})$.

Proof. The proof is similar to that of Theorem 2.3.4 except that the distortion bounds in (3.2.6) and (3.2.7) have to be used here in place of using the bounds in (1.2.44), (1.2.45) for the proof of Theorem 2.3.4. The details of proof are omitted. The function $l_\alpha(z)$ defined in (3.1.2) gives sharpness for (3.2.9) in U and for (3.2.10) in the annulus $r_2 \le 2/(1+\sqrt{5-4A})$, $A = \sqrt{1-1/R_2}$.

In the following result we find bounds on the growth of $|f(z)|$ for functions f in the class $CV_\alpha(R_1,R_2)$.

<u>Theorem</u> 3.2.4. <u>If</u> $f \in CV_{\alpha}(R_1, R_2)$, $R_2 < \infty$, $0 < \alpha < 1$, <u>then</u>

$$(3.2.11) \qquad \frac{(1+Ar)^{2\alpha-1} - 1}{A(2\alpha-1)} \le |f(z)| \le \frac{1-(1-Ar)^{2\alpha-1}}{A(2\alpha-1)}, \quad \alpha \ne 1/2,$$

$$(3.2.12) \qquad \frac{\ln(1+Ar)}{A} \le |f(z)| \le \frac{1}{A} \ln \frac{1}{1-Ar}, \quad \alpha = 1/2,$$

<u>where</u> $|z| = r$, <u>the left hand side inequalities hold in</u> (3.2.11), (3.2.12) <u>in the unit disc</u> U <u>and the right hand side inequalities in</u> (3.2.11), (3.2.12) <u>hold in the disc</u> $|z| \le 2/(1+\sqrt{5-4A})$, $A = \sqrt{1-1/R_2}$. <u>The inequalities</u> (3.2.11) <u>and</u> (3.2.12) <u>are sharp</u>.

<u>Proof</u>. By taking $z_1 = 0$ in Theorem 3.2.3, the inequalities (3.2.11), (3.2.12) follow for $z \ne 0$. For $z = 0$, inequalities (3.2.11), (3.2.12) are trivial. The function $l_{\alpha}(z)$ defined in (3.1.2) gives sharpness in (3.2.11) and (3.2.12).

<u>Corollary</u> 3.2.5. If $f \in CV_{\alpha}(R_1, R_2)$, $R_2 < \infty$, $0 < \alpha < 1$, then

$$(3.2.13) \qquad d \ge \frac{(1+A)^{2\alpha-1} - 1}{A(2\alpha-1)}, \qquad \alpha \ne 1/2$$

$$(3.2.14) \qquad d \ge \frac{\ln(1+A)}{A}, \qquad \alpha = 1/2$$

where, $d = \inf_{\zeta \in \partial f(U)} |\zeta|$, $A = \sqrt{1-1/R_2}$. The inequalities are sharp.

<u>Proof</u>. The corollary follows from Theorem 3.2.4. The functions $l_{\alpha}(z)$, defined in (3.1.2), give sharpness in the inequalities (3.2.13) and (3.2.14).

Remark. The Koebe domain for $CV_\alpha(R_1,R_2)$, $R_2 < \infty$, $0 < \alpha < 1$ is the disc

$$\{w : |w| < \frac{(1+A)^{2\alpha-1} - 1}{A(2\alpha-1)}\}, \qquad \alpha \neq 1/2$$

$$\{w : |w| < \frac{\ln\,(1+A)}{A}\} \quad , \qquad \alpha = 1/2$$

where, $A = \sqrt{1-1/R_2}$. This follows from Corollary 3.2.5.

The following result gives an integral operator that transforms functions in the class $ST(R_1,R_2)$ (cf. Section 1.2) into that in the class $CV_\alpha(R_1,R_2)$.

Proposition 3.2.1. Let $f \in \mathcal{A}_1$. Then, the function

$$f \in ST(R_1,R_2),$$

if and only if,

$$F(z) \equiv \int_0^z (\frac{f(\zeta)}{\zeta})^{1-\alpha}\, d\zeta \in CV_\alpha(R_1,R_2)$$

where, $0 \leq \alpha < 1$.

Proof. We have that the function $f \in ST(R_1,R_2)$, if and only if,

$$\int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta \in CV(R_1,R_2).$$

Equivalently,

$$F(z) \equiv \int_0^z (\frac{f(\zeta)}{\zeta})^{1-\alpha}\, d\zeta \in CV_\alpha(R_1,R_2)$$

by Theorem 3.2.1. This completes the proof of the proposition.

3.3 In this section an integral transform on the class $C_\alpha(K)$ (cf. Section 3.1) is studied. For functions in the class $C_\alpha(K)$, bounds on the moduli of second and third Taylor coefficients are found. A distortion theorem for the functions in the class $C_\alpha(K)$ and necessary and sufficient conditions for a function to be in the class $C_\alpha(K)$ are also obtained in this section. Finally, a rotation theorem is derived in this section for the class $CV_\alpha(R_1,R_2)$ (cf. Section 3.1).

Proposition 3.3.1. Let a function $f \in A$. Then $f \in C_\alpha(K)$, if and only if, $g(z) \equiv \int_0^z (f'(\tau))^{1/(1-\alpha)}\, d\tau \in C(K)$ where $\alpha < 1$ and $K > 0$.

Proof. We observe that $f(z)$ is locally univalent in U, if and only if, $g(z) \equiv \int_0^z (f'(\tau))^{1/(1-\alpha)}\, d\tau$ is locally univalent in U, where $\alpha < 1$. This, (3.2.2) and the definitions of the classes $C_\alpha(K)$ and $C(K)$ (cf. Section 1.2) together give the proposition.

Some of the applications of Proposition 3.3.1 are as follows. We begin with a lemma:

Lemma 3.3.1. Let $f \in C_\alpha(K)$ with $\alpha < 1$, $\alpha \neq 0$ and $K > 0$. Then, $k_\alpha(f;z)$ has no local minimum in $U\setminus\{0\}$ and $k_\alpha(f;z) > K$ in $U\setminus\{0\}$.

Proof. Proposition 3.3.1, equation (3.2.2) and (1.2.54) together give the lemma.

Remark. A direct proof of Lemma 3.3.1 without using Proposition 3.3.1 can be given as follows by using a technique of Peschl [99].

Let $k_\alpha(f;z_1)$ (cf. (3.1.1)), $0 < |z_1| < 1$ be a local minimum of $k_\alpha(f;z)$ in $U\setminus\{0\}$. Then,

$$0 = \frac{\partial}{\partial z} k_\alpha(f;z)\Big|_{z=z_1}$$

$$= \frac{1}{|z|\ |f'(z)|^{1/(1-\alpha)}\ 2(1-\alpha)} \left(\frac{zf''(z)}{f'(z)}\right)'$$

$$- \frac{1}{2z}\left(1 + \frac{zf''(z)}{(1-\alpha)f'(z)}\right)k_\alpha(f;z)\Big|_{z=z_1}.$$

Hence,

$$\frac{\partial^2}{\partial\bar{z}\partial z} k_\alpha(f;z_1) = -\left|\frac{1}{2z}\left(1 + \frac{zf''(z)}{(1-\alpha)f'(z)}\right)\right|^2 k_\alpha(f;z)\Big|_{z=z_1}. \tag{3.3.1}$$

For $z \in U\setminus\{0\}$, by the inequality (3.1.6) we have that $k_\alpha(f;z) > 0$. Expression (3.3.1) gives that $\partial^2 k_\alpha(f;z)/\partial\bar{z}\partial z < 0$ at $z = z_1$, a contradiction to the assumption that $k_\alpha(f;z)$ has a local minimum in $U\setminus\{0\}$ at $z = z_1$. Thus, the first assertion is proved. We have $\lim_{z\to 0} k_\alpha(f;z) = \infty$ and hence $k_\alpha(f;z)$ can not be constant in $U\setminus\{0\}$. This proves the lemma.

Theorem 3.3.1. *For an analytic and locally univalent function* f *in* U, $f \in C_\alpha(K)$, $\alpha < 1$, $\alpha \neq 0$, $K > 0$, *if and only if*, $k_\alpha(f;z) > K$ *in* $U\setminus\{0\}$.

Proof. Lemma 3.3.1 gives the 'only if' part. The 'if' part is easy to prove and is omitted.

Using Theorem 3.3.1, it can be seen that if $f \in C_\alpha(K)$, $0 < \alpha < 1$, $K > 0$ and $f_r(z) = f(rz)$, $0 < r < 1$, then the function $f_r \in C_\alpha(K^*) \subseteq C_\alpha(K)$ for a constant $K^* > K$.

A bound on the modulus of second Taylor coefficient for functions in the class $C_\alpha(K)$ is determined with the help of the following lemmas.

Lemma 3.3.2. Let $f \in C_\alpha(K)$ with $\alpha < 1$, $\alpha \neq 0$, $K > 0$ and $z_o \in U$. The following functional, defined in a neighbourhood of z_o

$$(3.3.2) \qquad u_\alpha(f;z) = \frac{1 - \left|\dfrac{f''(z)(1-|z|^2)}{f'(z)2(1-\alpha)} - \bar{z}\right|^2}{|f'(z)|^{1/(1-\alpha)}(1-|z|^2)}$$

has a local minimum at z_o, if and only if, $f(z)$ is of the form

$$(3.3.3) \qquad f(z) = \begin{cases} \dfrac{p}{q(cz+d)^{1-2\alpha}} + s \quad , & \text{for } \alpha \neq 1/2 \\[2ex] \dfrac{p}{c} \log (cz+d) \quad , & \text{for } \alpha = 1/2 \end{cases}$$

where, $p = (ad - bc)^{1-\alpha}$, $q = (2\alpha-1)c$, a,b,c,d,s being complex constants.

Proof. Proposition 3.3.1 and condition (3.1.10) give the lemma.

In the sequel, wherever reference to the function is not needed, the functional $u_\alpha(f;z)$ is denoted by $u_\alpha(z)$.

Lemma 3.3.3. Let $f \in C_\alpha(K)$, $\alpha < 1$, $\alpha \neq 0$, $K > 0$ and $z_o \in U$. When $f''(z_o)(1-|z_o|^2) \neq 2(1-\alpha)\bar{z}_o f'(z_o)$, the functional $u_\alpha(z)$ in (3.3.2) has a local maximum at z_o, if and only if, $f(z)$ is of the form (3.3.3).

Proof. Proposition 3.3.1 and condition (3.1.11) give the lemma.

Theorem 3.3.2. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k \in C_\alpha(K)$ *with* $\alpha < 1$, $\alpha \neq 0$, *and* $K > 0$. *Then, the inequality*

$$(3.3.4) \qquad \left|\frac{f''(z)(1-|z|^2)}{f'(z)2(1-\alpha)} - \bar{z}\right|^2 \leq 1-K\,|f'(z)|^{1/(1-\alpha)}(1-|z|^2)$$

is true for $z \in U$. *The inequality* (3.3.4) *is sharp*.

Proof. For the functional $u_\alpha(z)$ in (3.3.2) we have, for $z \in U$,

$$u_\alpha(z) \geq \liminf_{|z|\to 1} u_\alpha(z)$$

$$= \liminf_{|z|\to 1} \frac{1}{|f'(z)|^{1/(1-\alpha)}}\left[\operatorname{Re}\left(1 + \frac{zf''(z)}{(1-\alpha)f'(z)}\right) - \frac{(1-|z|^2)}{4(1-\alpha)^2}\left|\frac{f''(z)}{f'(z)}\right|^2\right]$$

$$= \liminf_{|z|\to 1} k_\alpha(z) \geq K$$

which gives (3.3.4).

Suppose that equality holds in (3.3.4) at $z_1 \in U$. Then $u_\alpha(z)$ has a local minimum at z_1. By lemmas 3.3.2 and 3.3.3 we

obtain that $f(z)$ is of the form (3.3.3) and $u_\alpha(z) \equiv K$ in U. The functional $u_\alpha(z) \equiv K$, if and only if,

$$(3.3.5)\quad f(z) = \begin{cases} \dfrac{e^{i\phi}(1-|a|^2)^{1-\alpha}}{K^{1-\alpha}\; a(2\alpha-1)(1+\bar{a}z)^{1-2\alpha}} + b \text{ or } \dfrac{e^{i\phi}}{K^{1-\alpha}} z+b; & \text{if } \alpha \neq 1/2 \\ \dfrac{e^{i\phi}}{a}\sqrt{\dfrac{1-|a|^2}{K}} \log(1+\bar{a}z) + b \text{ or } \dfrac{e^{i\phi}}{K^{1-\alpha}} z+b; & \text{if } \alpha = 1/2 \end{cases}$$

where, $a \in U\backslash\{0\}$, $b \in \mathbb{C}$ and $\phi \in \mathbb{R}$. Thus, Lemma 3.3.1 gives that equality for one z in U in inequality (3.3.4) implies equality for all z in U, and this happens, if and only if, $f(z)$ is of the form (3.3.5).

<u>Theorem</u> 3.3.3. <u>Let</u> $f(z) = \sum_{k=0}^{\infty} a_k z^k \in C_\alpha(K)$ <u>with</u> $\alpha < 1$, $\alpha \neq 0$ <u>and</u> $K > 0$. <u>Then</u>,

$$(3.3.6)\qquad \left|\frac{a_2}{a_1(1-\alpha)}\right|^2 \leq 1-K\,|a_1|^{1/(1-\alpha)}.$$

<u>The inequality is sharp only for the functions of the form</u> (3.3.5).

<u>Proof</u>. Taking $z = 0$ in Theorem 3.3.2, gives Theorem 3.3.3.

Next, a bound on the modulus of third Taylor coefficient for functions in the class $C_\alpha(K)$ is determined with the help of the following lemma.

<u>Lemma</u> 3.3.4. <u>Let</u> $f(z) = \sum_{k=0}^{\infty} a_k z^k \in C_\alpha(K)$ <u>with</u> $\alpha < 1$, $\alpha \neq 0$ <u>and</u>

K > 0. Then, the inequality

$$(3.3.7)\qquad \frac{1}{2}\left|\frac{1}{1-\alpha}\left(\frac{\alpha}{1-\alpha}\frac{(f''(z))^2}{f'(z)} + f'''(z)\right)\frac{1}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{(1-\alpha)f'(z)}\right)^2\right|$$

$$\times (1-|z|^2)^2(1 - \frac{K}{2}|f'(z)|^{1/(1-\alpha)}(1-|z|^2))$$

$$\leq 1 - \left|\frac{f''(z)(1-|z|^2)}{f'(z)2(1-\alpha)} - \bar{z}\right|^2 - K|f'(z)|^{1/(1-\alpha)}(1-|z|^2)$$

holds for $z \in U$. The inequality (3.3.7) is sharp only for the function $f(z)$ of the form (3.3.5).

Proof. Proposition 3.3.1 and the inequality (3.1.12) give the inequality (3.3.7). Now suppose that equality holds for some $z_1 \in U$ in (3.3.7). Then, the right hand side of (3.3.7) has to be zero at z_1. This means that equality is attained in (3.3.4) at z_1. Hence, $f(z)$ has to be of the form (3.3.5). Conversely, suppose that $f(z)$ is of the form (3.3.5). Then, both the sides are equal to zero, at all $z \in U$. This completes the proof of the lemma.

Theorem 3.3.4. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k \in C_\alpha(K)$ with $\alpha < 1$, $\alpha \neq 0$ and $K > 0$. Then,

$$(3.3.8)\qquad \left|\frac{a_3}{a_1}\right| \leq (1-K|a_1|^{1/(1-\alpha)})(1 - \frac{2}{3}\alpha)(1-\alpha).$$

Equality holds, if and only if, $f(z)$ is of the form (3.3.5).

Proof. For $z = 0$, inequality (3.3.7) gives that

$$(3.3.9) \qquad \frac{3}{1-\alpha}\left|\frac{a_3}{a_1} - \frac{1-\frac{2}{3}\alpha}{1-\alpha}\left(\frac{a_2}{a_1}\right)^2\right| \le \frac{1-\left|\frac{a_2}{a_1(1-\alpha)}\right|^2 - K|a_1|^{1/(1-\alpha)}}{1-\frac{K}{2}|a_1|^{1/(1-\alpha)}} .$$

From this by applying the triangle inequality and then inequality (3.3.6), the required inequality is obtained.

Remarks. (i) The class $CV_\alpha(R_1,R_2)$, $0 < \alpha < 1$, $R_2 < \infty$, being contained in $C_\alpha(1/R_2)$, the inequalities (3.3.6) and (3.3.8) continue to hold for functions in the class $CV_\alpha(R_1,R_2)$. Further, the function $l_\alpha(z)$ defined by (3.1.2) gives sharpness of (3.3.6) and (3.3.8) in $CV_\alpha(R_1,R_2)$.

(ii) For the class $C_\alpha(K)$, $\alpha < 1$, $\alpha \neq 0$, $K > 0$, Lemma 3.3.1 gives the analogue of the local minimum property (1.2.54) of $k(f;z)$ for functions f in the class $C(K)$ found by Wirths [159]. The inequalities (3.3.6), (3.3.7) and (3.3.8) for the class $C_\alpha(K)$,are analogous to the Wirths' inequalities (3.1.9), (3.1.12) and (3.1.13) respectively, found in [160], [159] for the class $C(K)$. Lemmas 3.3.2 and 3.3.3 for the class $C_\alpha(K)$ are similar to the Wirths' maximum-minimum principle for the class $C(K)$ determined in [159].

A bound on the modulus of fourth Taylor coefficient for functions in the class $CV_\alpha(R_1,R_2)$ is given by our following theorem:

<u>Theorem</u> 3.3.5. <u>If</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV_\alpha(R_1,R_2)$, $0 \le \alpha < 1$, <u>and</u> $R_2 < \infty$ <u>then</u>, for $K = 1/R_2$,

$$(3.3.10) \qquad |a_4| \le \frac{1}{\sqrt{2}} + \frac{2}{3}|a_2|\frac{(1-\alpha)^2(1-K)-|a_2|^2(1-\alpha(1-K/2))}{(1-\alpha)(1-K/2)} + |a_2|^3.$$

<u>Proof</u>. By (1.6.2) for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z/(1+\sum_{n=1}^{\infty} b_n z^n) \in S$, we obtain

$$2|b_3|^2 \le 1.$$

Equivalently,

$$|a_4 - 2a_2a_3 + a_2^3| \le \frac{1}{\sqrt{2}}.$$

Now, by the triangle inequality

$$(3.3.11) \qquad |a_4| \le \frac{1}{\sqrt{2}} + 2|a_2|\ |a_3 - a_2^2| + |a_2|^3.$$

Inequality (3.3.9) gives that for $f \in CV_\alpha(R_1,R_2) \subseteq C_\alpha(K)$,

$$(3.3.12) \qquad |a_3-a_2^2| \le \frac{(1-\alpha)^2(1-K) - |a_2|^2(1-\alpha(1-K/2))}{3(1-\alpha)(1-K/2)}.$$

Inequalities (3.3.11) and (3.3.12) give the inequality (3.3.10).

<u>Remarks</u>. (i) Since, by Lemma 2.4.1, the class $CVG(R_1,R_2) \subseteq CVG(0,R_2) = CV(0,R_2) = CV_0(0,R_2)$, by putting $\alpha = 0$ and $K = 1/R_2$ in the inequality (3.3.10),we obtain that if

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CVG(R_1, R_2)$, $R_2 < \infty$, then

$$|a_4| \le \frac{1}{\sqrt{2}} + \frac{2}{3}|a_2| \frac{1-|a_2|^2 - 1/R_2}{1-1/2R_2} + |a_2|^3. \tag{3.3.13}$$

(ii) For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CVG(R_1, R_2)$ with restricted $|a_2|$, and special R_2, the upper bound on $|a_4|$ in the inequality (3.3.13) is better than that in the inequality (1.2.38). For, R_2 satisfying $\sqrt{3} \le R_2 < \infty$ and $a_2 \in \mathbb{C}\setminus\{0\}$ with $|a_2| \le R_2^*$, where

$$R_2^* \equiv \left(\left(\frac{2}{3}\right)^3 - \frac{1}{\sqrt{2}} + \frac{\sqrt{R_2^2 - 1}}{2}\right)^{1/3} - \frac{2}{3} \le \left(1 - \frac{1}{R_2}\right)^{1/2}$$

the inequality (3.3.13) gives that

$$|a_4| \le \frac{1}{\sqrt{2}} + \frac{2}{3}|a_2| \frac{1-|a_2|^2 - 1/R_2}{1-1/2R_2} + |a_2|^3 \le (R_2^2 - 1)^{1/2}/2.$$

In the following, we obtain some results for the class $C_\alpha(K)$ that are analogues of the results of Wirths [160].

<u>Proposition</u> 3.3.2. Let $K > 0$, $\alpha < 1$, $\alpha \ne 0$, $f \in C_\alpha(K)$, be analytic and locally univalent in a disc $D_\epsilon = \{z : |z| < 1 + \epsilon\}$ for an $\epsilon > 0$ and $t : \{z : |z| = 1\} \longrightarrow \mathbb{R}$ be continuous and nonnegative. For any $a \in U$,

$$\left|\frac{1}{2\pi}\int_0^{2\pi} K[f'(e^{i\theta})]^{1/(1-\alpha)}(1-\bar{a}e^{i\theta})^2\, t(e^{i\theta})\, d\theta\right| \tag{3.3.14}$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(1 + \frac{e^{i\theta}f''(e^{i\theta})}{(1-\alpha)f'(e^{i\theta})}\right)|1-\bar{a}e^{i\theta}|^2\, t(e^{i\theta})\, d\theta$$

and equality occurs, if and only if, $f(z)$ is of the form (3.3.5).

Proof. We have

$$\left|\frac{1}{2\pi}\int_0^{2\pi} K[f'(e^{i\theta})]^{1/(1-\alpha)}(1-\bar{a}e^{i\theta})^2\, t(e^{i\theta})d\theta\right|$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} K|f'(e^{i\theta})|^{1/(1-\alpha)}\,|1-\bar{a}e^{i\theta}|^2\, t(e^{i\theta})\, d\theta.$$

Now the inequality (3.3.14) follows from Theorem 3.3.1. Shaprness follows from that of (3.1.7) with the help of Proposition 3.3.1.

Corollary 3.3.1. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k \in C_\alpha(K)$ with $\alpha < 1, \alpha \ne 0$, and $K > 0$. For any $a \in U\setminus\{0\}$ and any $b \in U$,

$$(3.3.15)\quad K\left|(f'(a))^{1/(1-\alpha)}(1-|a|^2)(1-\bar{b}a)\,\frac{a-b}{a} + (c_1)^{1/(1-\alpha)}\,\frac{b}{a}\right|$$

$$\le 1 + |b|^2 - \frac{2}{1-\alpha}\,\mathrm{Re}\left(b\,\frac{c_2}{c_1}\right).$$

Equality in (3.3.15) is attained, if $f(z)$ is of the form (3.3.5).

Proof. Choose, $t(z) = \dfrac{(1-\bar{b}z)(z-b)}{(1-\bar{a}z)(z-a)}$. Now, by evaluating the left hand side of (3.3.14), with $f_r(z) = f(rz)$ in place of $f(z)$ we have

$$L_a(f_r) \equiv \left|\frac{1}{2\pi}\int_0^{2\pi} K[f_r'(e^{i\theta})]^{1/(1-\alpha)}(1-\bar{a}e^{i\theta})^2\, t(e^{i\theta})d\theta\right|$$

$$= K\left|(rf'(ra))^{1/(1-\alpha)}(1-|a|^2)(1-\bar{b}a)\,\frac{a-b}{a} + r(f'(0))^{1/(1-\alpha)}\,\frac{b}{a}\right|$$

by the residue theorem. The right hand side of (3.3.14) with $f_r(z)$ in plce of $f(z)$ becomes

$$R_a(f_r) \equiv 1 + |b|^2 - \frac{2r}{1-\alpha} \operatorname{Re}(b \frac{c_2}{c_1}).$$

Now, letting r tend to 1 in the inequality $L_a(f_r) \leq R_a(f_r)$, the corollary is obtained.

<u>Theorem</u> 3.3.6. <u>Let</u> $f \in C_\alpha(K)$ <u>with</u> $\alpha < 1$, $\alpha \neq 0$, <u>and</u> $K > 0$. <u>Then</u>

(3.3.16) $$|f'(z)| \leq 1/(K(1-|z|^2))^{1-\alpha}, \qquad z \in U.$$

<u>Equality occurs in (3.3.16) for functions of the form (3.3.5).</u>

<u>Proof</u>. For $b = 0$, Corollary 3.3.1 gives Theorem 3.3.6.

<u>Corollary</u> 3.3.2. Let $f \in C_\alpha(K)$ and L_r be the length of $\gamma_r = \{f(re^{i\theta}) : 0 \leq \theta < 2\pi\}$ where $\alpha < 1$ and $K > 0$. Then,

(3.3.17) $$L_r \leq \frac{2\pi r}{(K(1-r^2))^{1-\alpha}} .$$

<u>Proof</u>. We have $L_r = r \int_0^{2\pi} |f'(re^{i\theta})| d\theta$. This and inequality (3.3.16) give inequality (3.3.17) for $\alpha < 1$, $\alpha \neq 0$. For $\alpha = 0$, it is known [160] that Theorem 3.3.6 holds. This observation and the formula for L_r give (3.3.17) for $\alpha = 0$. Thus the corollary is proved.

<u>Corollary</u> 3.3.3. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k \in C_\alpha(K)$ with $\alpha < 1$, $\alpha \neq 0$ and $K > 0$. Then,

$$(3.3.18) \qquad \left|\frac{c_2}{c_1(1-\alpha)}\right|^2 \le 1-K\,|c_1|^{1/(1-\alpha)}$$

Equality occurs in (3.3.18) for a function of the form (3.3.5).

<u>Proof</u>. Choose $a \in U\setminus\{0\}$ such that $\frac{ac_2}{c_1} \ge 0$ and $b = a$ in Corollary 3.3.1. Then, we have

$$K|c_1|^{1/(1-\alpha)} \le 1 + |a|^2 - \frac{2}{1-\alpha}\, a\, \frac{c_2}{c_1}.$$

Or, equivalently,

$$\left|\frac{c_2}{c_1(1-\alpha)}\right| \le \frac{1 + |a|^2 - K|c_1|^{1/(1-\alpha)}}{2|a|} \equiv G(|a|).$$

If, $1-K|c_1|^{1/(1-\alpha)} > 0$, then

$$\min_{0<|a|<1} G(|a|) = G((1-K|c_1|^{1/(1-\alpha)})^{1/2})$$

$$= (1-K|c_1|^{1/(1-\alpha)})^{1/2}.$$

If $K|c_1|^{1/(1-\alpha)} = 1$, then $G(|a|) = \frac{|a|}{2}$. This gives that $\left|\frac{c_2}{c_1}\right| = 0$. Thus, the corollary is proved.

In the following result a necessary and sufficient condition for a function to be in the class $C_\alpha(K)$ is determined.

Proposition 3.3.3. A function $f(z) = \sum_{k=0}^{\infty} c_k z^k \in C_\alpha(K)$, if and only if,

i) $f(\frac{z-a}{1-\bar{a}z}) \in C_\alpha(K)$, for any $a \in U$

ii) $|\frac{c_2}{c_1(1-\alpha)}|^2 \le 1-K\ |c_1|^{1/(1-\alpha)}$

where $\alpha < 1$, $\alpha \neq 0$ and $K > 0$.

Proof. Proposition 3.3.1 and condition (3.1.14) give the proposition.

Remark. The inequalities (3.3.14) and (3.3.18) for the class $C_\alpha(K)$, $\alpha < 1$, $\alpha \neq 0$, $K > 0$, are analogues of the Wirths' inequalities (3.1.7) and (3.1.9) respectively, for the class $C(K)$ determined in [160]. Proposition 3.3.3 for the class $C_\alpha(K)$ is analogous to the necessary and sufficient condition (3.1.14) for the class $C(K)$ obtained in [160].

A relation between the curvatures of the curves $f(|z| = r)$ and $g(|z| = r)$ is given by the following theorem when g is a special integral transform of a function f in $CV_\alpha(R_1.R_2)$.

Theorem 3.3.7. Let $f \in CV_\alpha(R_1,R_2)$ with $0 < \alpha < 1$, $R_2 < \infty$ and $g(z) \equiv \int_0^z (f'(\tau))^{1/(1-\alpha)}\, d\tau$. Then, the curvatures $k(f;z)$ and $k(g;z)$ for $|z| = r < 1$, $r > 0$ are related by

$$(3.3.19)\qquad \frac{(1-r)^{2\alpha}}{1-\alpha}(1-\alpha\ \frac{1+r}{1-r})k(f;z) \le k(g;z) \le \frac{(1+r)^{2\alpha}}{1-\alpha}(1-\alpha\ \frac{1-r}{1+r})k(f;z)$$

<u>Proof</u>. We have $f \in CV_\alpha(R_1,R_2) \subseteq CV$. Hence

$$\frac{1-r}{1+r} \le \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \le \frac{1+r}{1-r} \tag{3.3.20}$$

where $|z| = r < 1$. Further, we have

$$k(g;z) = \frac{\operatorname{Re}\left(1 + \dfrac{zf''(z)}{(1-\alpha)f'(z)}\right)}{\left|z[f'(z)]^{1/(1-\alpha)}\right|},$$

$$k(g;z) = \frac{1 - \dfrac{\alpha}{\operatorname{Re}\left(1 + \dfrac{zf''(z)}{f'(z)}\right)}}{(1-\alpha)|f'(z)|^{\alpha/(1-\alpha)}}\, k(f;z). \tag{3.3.21}$$

By the distortion theorem for the class CV(cf. Definition 1.2.3), we have

$$\frac{1}{(1+r)^2} \le |g'(z)| = |f'(z)|^{1/(1-\alpha)} \le \frac{1}{(1-r)^2},$$

equivalently,

$$\frac{1}{(1+r)^{2\alpha}} \le |f'(z)|^{\alpha/(1-\alpha)} \le \frac{1}{(1-r)^{2\alpha}}. \tag{3.3.22}$$

Now (3.3.22), together with (3.3.20) and (3.3.21) gives the theorem.

Finally, we obtain a rotation theorem for the class $CV_\alpha(R_1,R_2)$.

Theorem 3.3.8. If $f \in CV_\alpha(R_1, R_2)$, $R_2 < \infty$, $0 < \alpha < 1$, then

$$|\arg f'(z)| \leq (1-\alpha) \ln \frac{(1+Ar)^2}{1-r^2}, \qquad z \in U,$$

where $|z| = r$ and $A = \sqrt{1-1/R_2}$.

Proof. The proof of the theorem is similar to that of Theorem 2.4.3 except that the inequalities (3.3.4), (3.2.6) have to be used in place of (2.4.7), (2.4.1).

CHAPTER IV

RECIPROCAL COEFFICIENT REGIONS FOR CERTAIN CLASSES OF UNIVALENT POLYNOMIALS

4.1 Let p be a nonconstant polynomial, normalized by $p(0) = 0$, $p'(o) = 1$, and $p(z) \neq 0$ in the punctured unit disc $U\setminus\{0\}$. The polynomial $p(z)$ may be expressed as $p(z) = z/g(z)$ where $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ is analytic in the unit disc U. We denote $\psi(g) = z/g(z)$. Thus,

$$(4.1.1) \qquad p(z) = \psi(g) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}, \qquad z \in U.$$

Such a function g is uniquely determined and we call it the reciprocal of the polynomial p in U. For $\psi(g)$ varying over a certain class of polynomials of degree atmost n $(n \geq 2)$, the largest $(n-1)$-dimensional region formed by the ordered $(n-1)$-tuples $(b_1, b_2, \ldots, b_{n-1})$, if n is even; the ordered pairs (b_1, b_2), (b_1^2, b_2) if $n = 3$; and the ordered $(n-1)$-tuples $(b_1, b_2, \ldots, b_{n-1})$, $(b_1, \ldots, b^2_{(n-1)/2} \ldots, b_{n-1})$ if n is odd, $n \geq 5$; is called the reciprocal coefficient region of that class of polynomials.

It is known [17] that $p_2(z) = z + a_2 z^2$, a_2 real, is typically real in U, if and only if, it is univalent in U. Equivalently, p_2

is typically real in U, if and only if,

(4.1.2) $$|a_2| \le \frac{1}{2} .$$

It can be easily seen with the help of the inequality (4.1.2) that a normalized quadratic polynomial p_2 is typically real and univalent in U, if and only if,

$$b_1^n = b_n , \qquad n = 1,2,3 \ldots,$$

and

(4.1.3) $$b_1 \in \mathbb{E} \equiv \{x \in \mathbb{R}: |x| \le 1/2\},$$

where, the function $1 + \sum_{n=1}^{\infty} b_n z^n$ is the reciprocal of p_2 in U. Thus, the reciprocal coefficient region of the class of normalized typically real or univalent quadratic polynomials is the set $\mathbb{E}$, defined in (4.1.3).

In Section 4.2, necessary and sufficient conditions in terms of the reciprocal coefficient regions are determined for a polynomial $\psi(g)$ (cf. (4.1.1)) to be in the classes S_3^*, S_{2p-1}^*, St_3, CV_3 or CV_{2p-1}, $p \ge 4$ (cf. Section 1.5). Section 4.3 is devoted to the investigation of the same problem wherein we determine the reciprocal coefficient regions for the classes S_4, S_5, S_5^* or CV_5 (cf. Section 1.5).

4.2 In this section we determine necessary and sufficient conditions in terms of the reciprocal coefficient regions for a polynomial $\psi(g)$ (cf. (4.1.1)) to be in the classes S_3^*, S_{2p-1}^*, St_3,

CV_3 or CV_{2p-1}, $p \geq 4$, defined in Section 1.5.

We begin with characterizing the reciprocal coefficient region for the class S_3^*.

Define (Figures 4.2.1 and 4.2.2),

$$\Omega_1 = \{(x,y) \in \mathbb{R}^2 : y \leq (x \pm 1/3)^2 + 2/9,\ -1/3 \leq x^2 - y \leq 1/5\},$$

$$\Omega_2 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x,\ 71x^2 - 167xy + 96y^2 - 23x + 32y \leq 0,$$
$$1/5 \leq x - y \leq 1/3\}.$$

<u>Theorem</u> 4.2.1 <u>For the function</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, b_1 <u>and</u> b_2 <u>real</u>, $\psi(g)$ <u>is in the class</u> S_3^*, <u>if and only if</u>,

$$(4.2.1) \qquad b_1 b_{n-1} - b_n = (b_1^2 - b_2) b_{n-2}, \qquad n = 3,4,5,\ \ldots$$

<u>and</u>

<u>either</u> $(b_1,\ b_2) \in \Omega_1$ <u>or</u> $(b_1^2,\ b_2) \in \Omega_2$.

<u>Consequently, the reciprocal coefficient region of</u> S_3^* <u>is</u> $\Omega_1 \cup \Omega_2$.

<u>Proof</u>. Let $\psi(g) = z/g(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n) = z + a_2 z^2 + a_3 z^3 \in S_3^*$. For $n \geq 1$, we have

$$b_n = - \sum_{k=o}^{n-1} b_k\, a_{n-k+1}$$

where $a_k = 0$ for $k \geq 4$ and $b_o = 1$. Therefore,

$$b_1 = -a_2,\ b_2 = a_2^2 - a_3$$

and, for $n \geq 3$,

$$b_n = -(b_{n-2}a_3 + b_{n-1}a_2)$$

$$= -b_{n-2}\,(b_1^2-b_2) + b_{n-1}b_1 \ .$$

Thus the equation (4.2.1) is established.

Since $\psi(g) = z+a_2z^2+a_3z^3 \in S_3^*$, we get (cf. (1.5.3)) that

$$|a_2| \le \begin{cases} \dfrac{1+3a_3}{2} & \text{when } -\frac{1}{3} \le a_3 \le \frac{1}{5} \\[2ex] 4\left(\dfrac{(1-3a_3)2a_3}{9-25a_3}\right)^{1/2}, & \text{when } \frac{1}{5} \le a_3 \le \frac{1}{3}. \end{cases}$$

Substituting the values of a_2 and a_3 in terms of b_1 and b_2, we have

$$(4.2.2) \qquad |b_1| \le \frac{1+3(b_1^2-b_2)}{2} \qquad , \text{ if } -\frac{1}{3} \le b_1^2-b_2 \le \frac{1}{5}$$

$$(4.2.3) \qquad |b_1| \le 4\left(\frac{(1-3(b_1^2-b_2))2(b_1^2-b_2)}{9-25(b_1^2-b_2)}\right)^{1/2}, \text{ if } \frac{1}{5} \le b_1^2-b_2 \le \frac{1}{3}.$$

For $b_1 \ge 0$, the inequality (4.2.2) is equivalent to

$$b_2 \le (b_1-1/3)^2 + 2/9, \qquad \text{when } -1/3 \le b_1^2-b_2 \le 1/5.$$

Further,

$$(b_1-1/3)^2 + \frac{2}{9} \le (b+1/3)^2 + \frac{2}{9} \ .$$

Thus we obtain that the point $(b_1, b_2) \in \Omega_1$. The case $b_1 < 0$ is similarly handled.

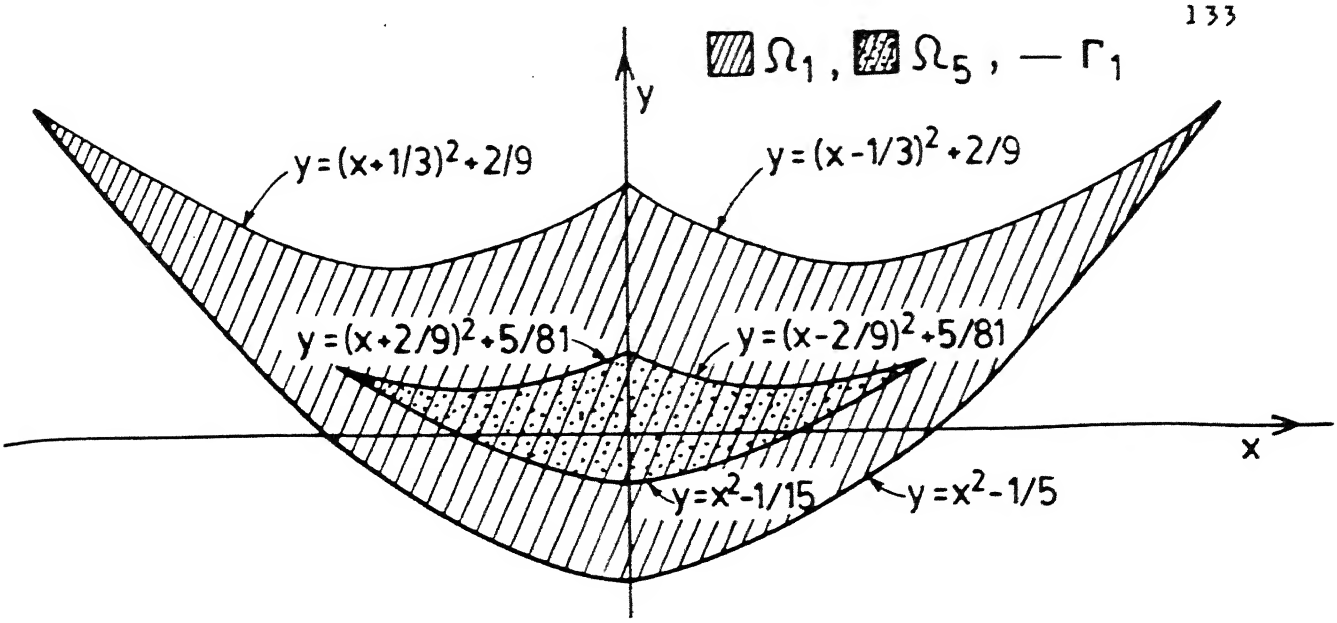

Fig. 4.2.1

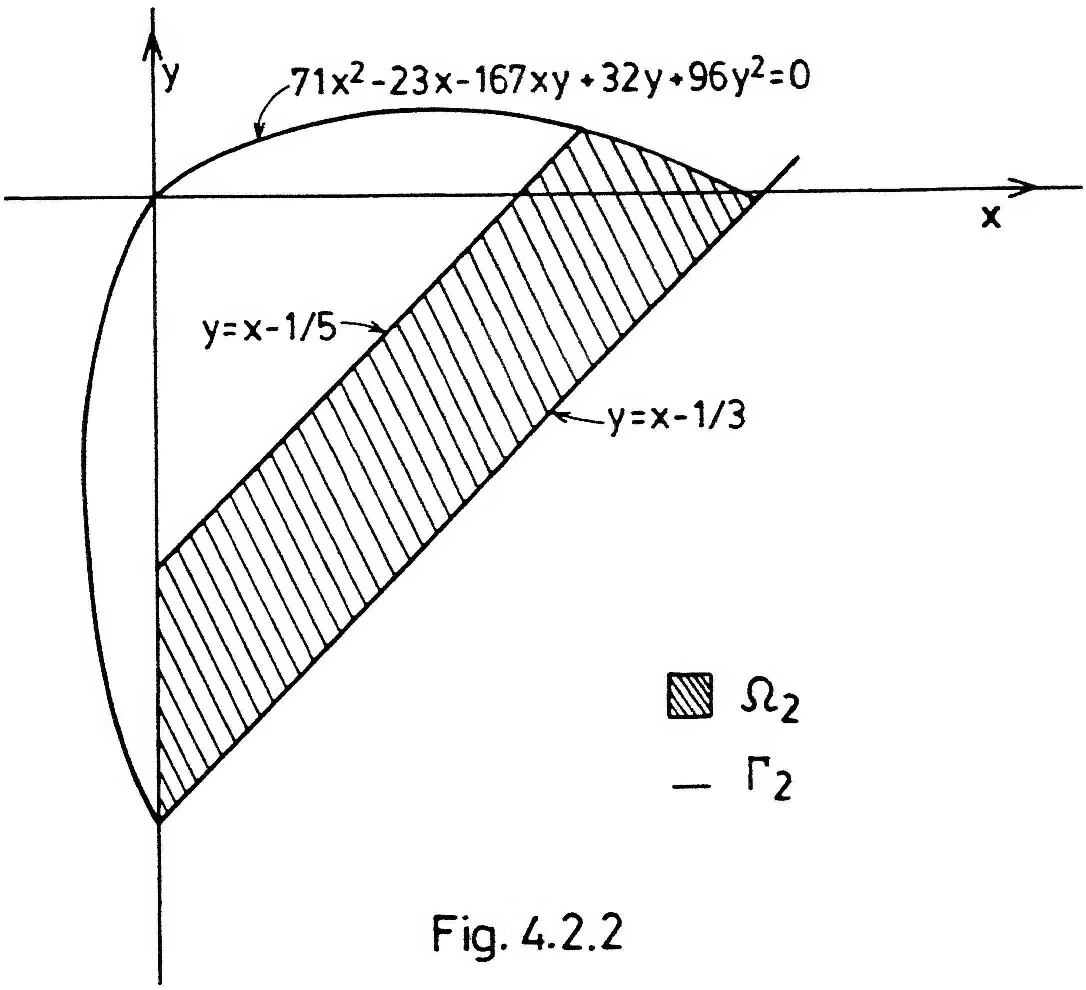

Fig. 4.2.2

The inequality (4.2.3) is equivalent to

$$71b_1^4 - 23b_1^2 - 167b_1^2 b_2 + 96b_2^2 + 32b_2 \le 0, \text{ if } 1/5 \le b_1^2 - b_2 \le 1/3.$$

This implies that the point $(b_1^2, b_2) \in \Omega_2$.

Conversely, let the equation (4.2.1) hold and either (b_1, b_2) be in Ω_1 or $(b_1^2, b_2) \in \Omega_2$ for $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, $z \in U$. Equation (4.2.1) gives that $\psi(g) = z/g(z)$ is a cubic polynomial and the condition $(b_1, b_2) \in \Omega_1$ or $(b_1^2, b_2) \in \Omega_2$ implies that $\psi(g) \in S_3^*$ (cf. (1.5.3)). Hence the theorem is proved.

<u>Remark</u>. It is easy to verify that the equations (1.5.6) and (4.2.1) in fact are the same. Further, the regions Ω_1, Ω_2 in Theorem 4.2.1 and the regions D_1, D_3 determined by Silverman and Silvia [139] (cf. Section 1.5) are related by

$$\Omega_1 = D_1 \text{ and } \quad \Omega_2 \subsetneqq D_3.$$

Thus, the reciprocal coefficient region $\Omega_1 \cup \Omega_2$ of S_3^* is an improvement over the reciprocal coefficient region $D_1 \cup D_3$ of the class of normalized univalent cubic polynomials in U found in [139].

<u>Examples</u>. It follows from Theorem 4.2.1 that the functions $z/(1 + \sum_{n=1}^{\infty} z^{2n}/9^n)$, $z/(1 + \sum_{n=1}^{\infty} z^n/4^n)$, and $z/(1 + \sum_{n=1}^{\infty} z^{2n}/(-5)^n)$, are in the class S_3^*.

Next we determine necessary and sufficient conditions in terms of the reciprocal coefficient region for a polynomial $\psi(g)$ (cf. (4.1.1)) to be in the class S_{2p-1}^*, $p \ge 4$ (cf. Section 1.5).

Define, for $p \geq 4$,

$$A_1 = \{(x_1, x_2, x_3, \ldots, x_{2p-2}) \in \mathbb{R}^{2p-2}: x_1 = x_2 = \ldots = x_{p-2} = x_p = x_{p+1} = \ldots = x_{2p-3} = 0, \; x_{p-1}, x_{2p-2} \text{ are arbitrary } \},$$

$$A_2 = \{(x_1, x_2, x_3, \ldots, x_{2p-2}) \in \mathbb{R}^{2p-2}: [x_{p-1} + \frac{p}{2(2p-1)}]^2 \geq \frac{(p-4)^2-12}{4(2p-1)^2} + x_{2p-2} \leq [x_{p-1} - \frac{p}{2(2p-1)}]^2\},$$

$$A_3 = \{(x_1, x_2, x_3, \ldots, x_{2p-2}) \in \mathbb{R}^{2p-2}: \frac{p+1}{(2p-1)(3p-1)} \geq x_{p-1}^2 - x_{2p-2} \geq -(2p-1)^{-1}\},$$

where throughout in the sequel for a, b, c real, by $a \geq b \leq c$ we mean that $a \geq b$ and $b \leq c$.

Set

$$\Omega_3 = \bigcap_{i=1}^{3} A_i .$$

Further, define for $p \geq 4$,

$$A_4 = \{(x_1, x_2, x_3, \ldots, x_{2p-2}) \in \mathbb{R}^{2p-2}: x_{p-1} \geq 0 \}$$

$$A_5 = \{(x_1, x_2, x_3, \ldots, x_{2p-2}) \in \mathbb{R}^{2p-2}: (p+1)(2p-1)^{-1}(3p-1)^{-1} \leq x_{p-1} - x_{2p-2} \leq (2p-1)^{-1}\}$$

$$A_6 = \{(x_1, x_2, x_3, \ldots, x_{2p-2}) \in \mathbb{R}^{2p-2} : (23p^2-10p-1)\, x_{p-1}^2$$

$$- (55p^2-26p-1)\, x_{p-1}x_{2p-2} + 16p(2p-1)x_{2p-2}^2$$

$$+ (p^2-14p+1)x_{p-1}+16px_{2p-2} \le 0\}$$

and set

$$\Omega_4 = A_1 \cap \bigcap_{i=4}^{6} A_i$$

<u>Theorem</u> 4.2.2 <u>Let</u> $p \ge 4$. <u>For the function</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>with</u> b_n <u>real</u> $1 \le n \le 2p-2$, $\psi(g)$ <u>is in the class</u> S^*_{2p-1}, <u>if and only if</u>,

$$(4.2.4) \qquad b_{p-1}\, b_{n-p+1} - b_n = (b_{p-1}^2 - b_{2p-2})b_{n-2p+2}, \qquad n \ge 2p-1,$$

<u>and</u>

<u>either</u>, $\quad (b_1, b_2, b_3, \ldots, b_{p-1}, \ldots, b_{2p-2}) \in \Omega_3$

<u>or</u>, $\quad (b_1, b_2, b_3, \ldots, b_{p-2}, b_{p-1}^2, b_p, \ldots, b_{2p-2}) \in \Omega_4$.

<u>Consequently, the reciprocal coefficient region of</u> S^*_{2p-1} <u>is</u> $\Omega_3 \cup \Omega_4$.

<u>Proof</u>. Let $\psi(g) = z + a_p z^p + a_{2p-1} z^{2p-1} \in S^*_{2p-1}$ for $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ with b_k real, $1 \le k \le 2p-2$. For $n \ge 1$,

$$b_n = -\sum_{k=1}^{n} b_{n-k}\, a_{k+1}$$

where $\quad a_2 = a_3 = \ldots = a_{p-1} = a_{p+1} = a_{p+2} = \ldots = a_{2p-2} = a_k = 0$, $k \ge 2p$ and $b_o = 1$. Therefore,

$$b_1 = 0.$$

Assume that $b_n = 0$ for $1 \le n \le m$, $m \in \mathbb{N}$, $1 \le m \le p-3$. We have

$$b_{m+1} = 0$$

because $a_2 = \ldots = a_{p-1} = 0$. Therefore,

$$b_1 = b_2 = .. = b_{p-2} = 0.$$

$$b_{p-1} = -a_p.$$

$$b_p = -(b_1 a_p + b_o a_{p+1}) = 0.$$

Assume that $b_j = 0$ for $p \le j \le 2p-4$, j is an integer. We have

$$b_{j+1} = -b_{j+2-p}\, a_p = 0$$

because $2 \le j+2-p \le p-2$. Hence,

$$b_p = b_{p+1} = .. = b_{2p-3} = 0.$$

$$b_{2p-2} = -(b_{p-1}a_p + b_o a_{2p-1})$$

$$= -\,(-a_p^2 + a_{2p-1}) = a_p^2 - a_{2p-1}.$$

For $n \ge 2p-1$,

$$b_n = -(b_{n-p+1}a_p + b_{n-2p+2}\, a_{2p-1})$$

$$= -\,(-b_{n-p+1}b_{p-1} + b_{n-2p+2}\,(a_p^2 - b_{2p-2})).$$

This implies the equation (4.2.4).

By the inequality (1.5.3),

$$(4.2.5) \qquad |b_{p-1}| \le \frac{1+(2p-1)(b_{p-1}^2-b_{2p-2})}{p},$$

$$\text{if } -(2p-1)^{-1} \le b_{p-1}^2-b_{2p-2} \le (p+1)(2p-1)^{-1}(3p-1)^{-1}$$

$$(4.2.6) \quad |b_{p-1}| \le 4\sqrt{\frac{\{1-(2p-1)(b_{p-1}^2-b_{2p-2})\}\ p(b_{p-1}^2-b_{2p-2})}{(p+1)^2-(3p-1)^2(b_{p-1}^2-b_{2p-2})}}\ ,$$

$$\text{if } (p+1)(2p-1)^{-1}(3p-1)^{-1} \le b_{p-1}^2-b_{2p-2} \le (2p-1)^{-1}.$$

For $b_{p-1} \ge 0$ the inequality (4.2.5) is equivalent to

$$b_{2p-2} + \frac{p^2-4(2p-1)}{4(2p-1)^2} \le [\ b_{p-1} - \frac{p}{2(2p-1)}\ ]^2$$

provided $-(2p-1)^{-1} \le b_{p-1}^2 - b_{2p-2} \le (p+1)\ (2p-1)^{-1}(3p-1)^{-1}$.

This implies that $(b_1,\ b_2,\ b_3,\ \ldots,\ b_{2p-2}) \in \Omega_3$ because

$$\{\ b_{p-1} - \frac{p}{2(2p-1)}\ \}^2 \le \{\ b_{p-1} + \frac{p}{2(2p-1)}\ \}^2.$$

Similarly, the case $b_{p-1} \le 0$ can be handled.

The inequality (4.2.6) is equivalent to

$$[-(3p-1)^2 + 16p(2p-1)]\ b_{p-1}^4 + \{(p+1)^2-16p\}\ b_{p-1}^2 + \{(3p-1)^2$$

$$-\ 32p(2p-1)\}\ b_{p-1}^2 b_{2p-2} + 16pb_{2p-2} + 16p(2p-1)\ b_{2p-2}^2 \le 0.$$

This implies that $(b_1, b_2, \ldots, b_{p-2}, b_{p-1}^2, b_p, \ldots, b_{2p-2}) \in \Omega_4$.

Conversely, let $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, $z \in U$, its coefficients satisfy the equation (4.2.4) and either $(b_1, b_2, \ldots, b_{2p-2}) \in \Omega_3$ or $(b_1, b_2, \ldots, b_{p-2}, b_{p-1}^2, b_p, \ldots, b_{2p-2}) \in \Omega_4$. Set $\psi(g) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$. For $n \geq 1$,

$$a_{n+1} = -\sum_{k=0}^{n-1} a_{1+k}\, b_{n-k}.$$

Therefore, for $2 \leq n \leq p-1$,

$$a_n = 0,\ a_p = -b_{p-1},$$

and for $p+1 \leq m \leq 2p-2$,

$$a_m = 0.$$

$$a_{2p-1} = -(b_{2p-2} - b_{p-1}^2) = b_{p-1}^2 - b_{2p-2}.$$

$$\begin{aligned} a_{2p} &= -\,(b_{2p-1} - b_{p-1}\, b_p + a_{2p-1} b_1) \\ &= -\,\{b_{p-1} b_p - (b_{p-1}^2 - b_{2p-2})\, b_1 - b_{p-1} b_p + (b_{p-1}^2 - b_{2p-2}) b_1\} \\ &= 0 \end{aligned}$$

by the equation (4.2.4). Assume that $a_j = 0$ for $2p \leq j \leq q$, q is an integer, $q \geq 2p$.

$$a_{q+1} = -(b_q + a_p\, b_{q-p+1} + a_{2p-1}\, b_{q-2p+2})$$

$$= -(b_{p-1}\, b_{q-p+1} - (b_{p-1}^2 - b_{2p-2})\, b_{q-2p+2} - b_{p-1} b_{q-p+1}$$

$$+ (b_{p-1}^2 - b_{2p-2})\, b_{q-2p+2})$$

$$= 0$$

by the equation (4.2.4). Therefore, $\psi(g)$ is of the form

$$\psi(g) = z + a_p z^p + a_{2p-1} z^{2p-1}.$$

Either $(b_1, \ldots, b_{2p-2}) \in \Omega_3$ or $(b_1, b_2, \ldots, b_{p-2}, b_{p-1}^2, b_p \ldots b_{2p-2})$ is in Ω_4 implies that $\psi(g) \in S^*_{2p-1}$, by the inequality (1.5.3). Hence the theorem is proved.

Next we determine necessary and sufficient conditions in terms of the reciprocal coefficient region for a polynomial $\psi(g)$ (cf. (4.1.1)) to be in the class St_3 (cf. Section 1.5).

Define (Figures 4.2.1 and 4.2.2),

$$\Gamma_1 = \{(x,y) \in \mathbb{R}^2 : y = (x \pm 1/3)^2 + 2/9,\ -1/3 \le x^2 - y \le 1/5\},$$

$$\Gamma_2 = \{(x,y) \in \mathbb{R}^2 : 0 \le x,\ 71x^2 - 167xy + 96y^2 - 23x + 32y = 0,$$
$$1/5 \le x - y \le 1/3\}.$$

<u>Theorem</u> 4.2.3 <u>For the function</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>with</u> b_1 <u>and</u> b_2 <u>real</u>, $\psi(g)$ <u>is in</u> St_3, <u>if and only if, the equation</u> (4.2.1) <u>holds and</u>

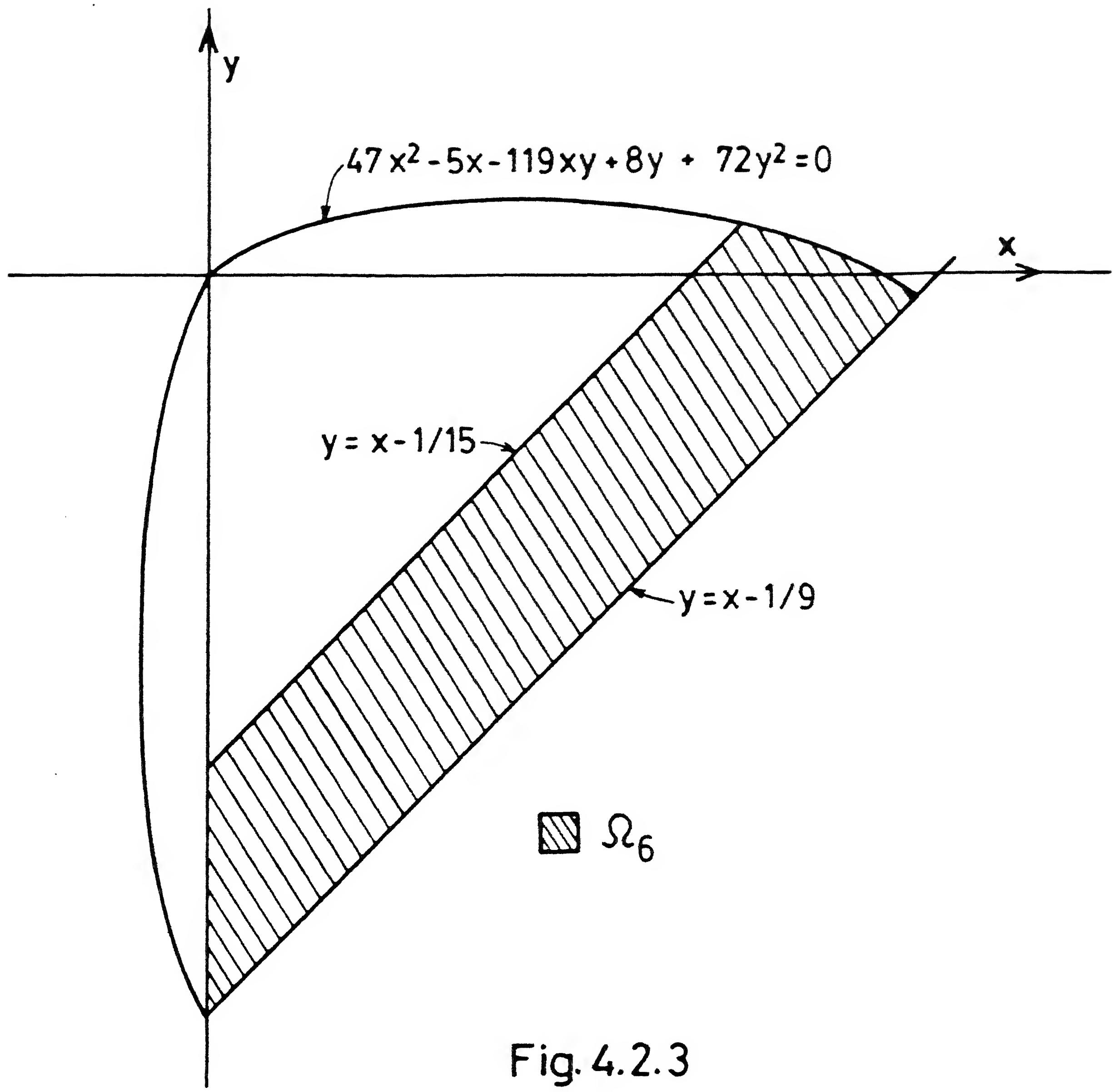

Fig. 4.2.3

<u>either</u> $(b_1, b_2) \in \Gamma_1$ <u>or</u> $(b_1^2, b_2) \in \Gamma_2$.

<u>Consequently, the reciprocal coefficient region of</u> St_3 <u>is</u> $\Gamma_1 \cup \Gamma_2$.

<u>Proof</u>. The proof is similar to that of Theorem 4.2.1 except that the equations (1.5.4) have to be used in place of the inequality (1.5.3). The details of the proof are therefore omitted.

<u>Remark</u>. It can be seen that the curves Γ_1 and Γ_2 in Theorem 4.2.3 are contained in the regions Ω_1 and Ω_2 in Theorem 4.2.1 respectively.

<u>Examples</u>. It follows from Theorem 4.2.3 that the functions

$z/(1 + \sqrt{23/71}\, z + \sum_{n=0}^{\infty} (-1)^n ((\sqrt{23/71}\, z)^{3n+3} + (\sqrt{23/71}\, z)^{3n+4}))$

and $z/(1+ \sum_{n=1}^{\infty} z^{2n}/(\pm 3)^n)$ are in the class St_3.

In the following result necessary and sufficient conditions in terms of the reciprocal coefficient region for a polynomial $\psi(g)$ to be in the class CV_3 (cf. Section 1.5) are determined.

Define (Figures 4.2.1, 4.2.3)

$$\Omega_5 = \{(x,y) \in \mathbb{R}^2 : x^2-15^{-1} \le y \le x^2+9^{-1},\ (x+2/9)^2 \ge y-5/81 \le (x - 2/9)^2\},$$

Ω_6 = the closure of the domain lying in $x \ge 0$, bounded by $y = x-1/15$, $y = x-1/9$ and $47x^2 - 119xy + 72y^2 - 5x + 8y = 0$.

<u>Theorem</u> 4.2.4 <u>For the function</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>with</u> b_1 <u>and</u> b_2 <u>real</u>, $\psi(g)$ <u>is in the class</u> CV_3, <u>if and only if, the equation</u> (4.2.1) <u>holds and</u>

$$\text{\underline{either} } (b_1, b_2) \in \Omega_5 \text{ \underline{or} } (b_1^2, b_2) \in \Omega_6.$$

<u>Consequently, the reciprocal coefficient region of</u> CV_3 <u>is</u> $\Omega_5 \cup \Omega_6$.

<u>Proof</u>. Alexander's type of relation between the classes CV_3, S_3^* and the method of proof of Theorem 4.2.1 give the assertion.

<u>Remark</u>. It can be verified that the regions Ω_5 in Theorem 4.2.4 and Ω_1 in Theorem 4.2.1 are related by

$$\Omega_5 \subseteq \Omega_1.$$

<u>Examples</u>. It follows from Theorem 4.2.4 that the functions

$z/(1 + \sum_{n=1}^{\infty} z^{2n}/10^n)$, $z/(1 + \sum_{n=1}^{\infty} z^n/4^n)$, $z/(1 + \sum_{n=1}^{\infty} z^{2n}/(-5)^n)$

$z/(1 + \sqrt{5/47}\, z + \sum_{n=0}^{\infty} (-1)^n ((\sqrt{5/47}\, z)^{3n+3} + (\sqrt{5/47}\, z)^{3n+4}))$ and

$z/(1 + \sum_{n=1}^{\infty} z^{2n}/(-9)^n)$ are in the class CV_3.

Next we determine necessary and sufficient conditions in terms of the reciprocal coefficient region for a polynomial $\psi(g)$ (cf. (4.1.1)) to be in the class CV_{2p-1}, $p \geq 4$ (cf. Section 1.5).

Define

$$A_7 = \{(x_1, x_2, x_3, \ldots, x_{2p-2}) \in \mathbb{R}^{2p-2}: [x_{p-1}-(p/\sqrt{2}(2p-1))^2]^2$$

$$\geq (p^4-4(2p-1)^2)/4(2p-1)^4 + x_{2p-2}$$

$$\leq [x_{p-1}+(p/\sqrt{2}(2p-1))^2]^2\},$$

$$A_8 = \{(x_1, x_2, x_3, \ldots, x_{2p-2}) \in \mathbb{R}^{2p-2}: (p+1)(2p-1)^{-2}(3p-1)^{-1}$$

$$\geq x_{p-1}^2 - x_{2p-2} \geq -(2p-1)^{-2}\}$$

$$\Omega_7 = A_1 \cap \bigcap_{i=7}^{8} A_i$$

$$A_9 = \{(x_1, x_2, x_3, \ldots, x_{2p-2}) \in \mathbb{R}^{2p-2}: (p+1)(2p-1)^{-2}(3p-1)^{-1}$$

$$\leq x_{p-1} - x_{2p-2} \leq (2p-1)^{-2}\}$$

$$A_{10} = \{(x_1, x_2, x_3, \ldots, x_{2p-2}) \in \mathbb{R}^{2p-2}: (-9p^3+70p^2-65p+16)x_{p-1}^2$$

$$+ (9p^3-134p^2+129p-32)x_{p-1}x_{2p-2} + 16(2p-1)^2 x_{2p-2}^2$$

$$+ (p^3+2p^2-31p+16)\, x_{p-1}/(2p-1) + 16x_{2p-2} \leq 0\},$$

and

$$\Omega_8 = A_1 \cap A_4 \cap \bigcap_{i=9}^{10} A_i.$$

<u>Theorem</u> 4.2.5 <u>Let</u> $p \geq 4$. <u>For the function</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>with</u> b_n <u>real</u>, $1 \leq n \leq 2p-2$, <u>the function</u> $\psi(g)$ <u>is in the class</u> CV_{2p-1}, <u>if and only if</u>, <u>the equation</u> (4.2.4) <u>holds and</u>

<u>either</u> $(b_1, b_2, b_3, \dots, b_{2p-2}) \in \Omega_7$

<u>or</u> $(b_1, b_2, \dots, b_{p-2}, b_{p-1}^2, b_p, \dots, b_{2p-2}) \in \Omega_8$.

<u>Consequently, the reciprocal coefficient region of</u> CV_{2p-1} <u>is</u> $\Omega_7 \cup \Omega_8$.

<u>Proof</u>. Alexander's type of relation between the classes CV_{2p-1}, S^*_{2p-1} and the method of proof of Theorem 4.2.2 make the assertion of Theorem 4.2.5.

<u>Remark</u>. It can be observed that the regions Ω_3 in Theorem 4.2.2 and Ω_7 in Theorem 4.2.5 are related by

$$\Omega_7 \subseteq \Omega_3 .$$

4.3 In this section necessary and sufficient conditions are determined in terms of the reciprocal coefficient regions for a polynomial $\psi(g)$ (cf. (4.1.1)) to be in the classes S_4, S_5, S^*_5 or CV_5 (cf. Section 1.5).

We begin with the characterization of the reciprocal coefficient region of the class S_4.

Define

$$\Omega_9 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -3(\sqrt{5} + 1)8^{-1} \le x_1 \le 3(\sqrt{5} + 1)8^{-1},$$
$$x_2 = (x_1+3^{-1})^2 - 9^{-1},\ x_1^3 - 2x_1x_2 + x_3 = -4^{-1}\}.$$

<u>Theorem</u> 4.3.1 <u>For the function</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>with</u> b_1, b_2 <u>and</u> b_3 <u>real</u>, $\psi(g)$ <u>is in the class</u> S_4, <u>if and only if</u>,

$$(4.3.1) \qquad b_1 b_{n-1} - b_n = (b_1^2 - b_2) b_{n-2} + b_{n-3}\, 4^{-1}, \qquad n = 4,5,6,\dots,$$

<u>and</u>

$$(b_1, b_2, b_3) \in \Omega_9.$$

<u>Consequently, the reciprocal coefficient region of</u> S_4 <u>is</u> Ω_9.

<u>Proof</u>. Let $\psi(g) = z+a_2z^2+a_3z^3+4^{-1}z^4 \in S_4$. For $n \ge 1$,

$$b_n = - \sum_{k=0}^{n-1} b_k\, a_{n-k+1}$$

where $a_k = 0$ for $k \ge 5$ and $b_o = 1$. Hence

$$b_1 = -a_2, \; b_2 = a_2^2 - a_3,$$

$$b_3 = -4^{-1} - b_1^3 + 2b_1b_2.$$

For $n \ge 4$,

$$b_n = -(b_{n-3}\, a_4 + b_{n-2}\, a_3 + b_{n-1}\, a_2)$$

$$= -4^{-1}\, b_{n-3} - b_{n-2}\, (b_1^2 - b_2) + b_{n-1}\, b_1.$$

This equation gives the equation (4.3.1).

Next we show that the point $(b_1, b_2, b_3) \in \Omega_9$. Since $\psi(g) = z + a_2 z^2 + a_3z^3 + 4^{-1} z^4 \in S_4$ we get from the necessary condition [16] for S_4 given in Section 1.5 that

$$3(b_1^2 - b_2) = -2b_1, \text{ when } -3(\sqrt{5} + 1)8^{-1} \le -b_1 \le 3(\sqrt{5} + 1)8^{-1}.$$

The last equation is equivalent to

$$b_2 = (b_1 + 3^{-1})^2 - 9^{-1},$$

which along with the last inequality implies that the point $(b_1, b_2, b_3) \in \Omega_9$.

Conversely, let $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, $z \in U$ satisfy the equation (4.3.1) and the point $(b_1, b_2, b_3) \in \Omega_9$. Say,

$$\psi(g) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U.$$

For $n \ge 1$,

$$a_{n+1} = -\sum_{k=0}^{n-1} a_{k+1} b_{n-k}.$$

Hence,

$$a_2 = -b_1, \; a_3 = b_1^2 - b_2,$$

$$a_4 = -b_3 + 2b_1b_2 - b_1^3 = 4^{-1}$$

and

$$a_5 = -(b_4 - b_1b_3 + b_1^2b_2 - b_2^2 + b_1 4^{-1}) = 0$$

by equation (4.3.1). Assume that $a_{n+1} = 0$ for $4 \le n \le p$, p is an integer, $p \ge 4$. Then, by equation (4.3.1)

$$a_{p+2} = -(a_1\, b_{p+1} + a_2\, b_p + a_3\, b_{p-1} + a_4\, b_{p-2})$$

$$= -(b_{p+1} - b_1\, b_p + (b_1^2 - b_2)\, b_{p-1} + b_{p-2}\, 4^{-1})$$

$$= 0.$$

Hence, $\psi(g)$ is of the form

$$\psi(g) = z + a_2\, z^2 + a_3\, z^3 + 4^{-1}\, z^4.$$

Now combining the sufficient condition [16] for S_4 given in Section 1.5 and the relation $(b_1, b_2, b_3) \in \Omega_9$, we obtain that $\psi(g) \in S_4$. This proves the theorem.

<u>Example</u>. It follows from Theorem 4.3.1 that the function $z/(1 + \sum_{n=1}^{\infty} z^{3n}/(-4)^n)$ is in the class S_4.

Next necessary and sufficient conditions are determined in terms of the reciprocal coefficient region for a polynomial $\psi(g)$ (cf. (4.1.1)) to be in the class S_5 (cf. Section 1.5).

Define

$$\Omega_{10} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 :\ x_1=x_3=0,\ (x_2+3/10)^2+11/100 \geq x_4 \\ \leq (x_2-3/10)^2 + 11/100,\ 0 \leq x_2^2 - x_4 \leq 1/10\},$$

$$\Omega_{11} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3 = 0,\ 3x_2^4 + 2x_2^3 - 3x_2^2 - 10x_2^2x_4 - 2x_2x_4 + 5x_4^2 + 4x_4 \le 0,$$

$$3x_2^4 - 2x_2^3 - 3x_2^2 - 10x_2^2x_4 + 2x_2x_4 + 5x_4^2 + 4x_4 \le 0,$$

$$1/10 \le x_2^2 - x_4 \le 1/5\ \}.$$

Theorem 4.3.2 *For the function* $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *with* b_n *real*, $1 \le n \le 4$, $\psi(g)$ *is in the class* S_5, *if and only if*,

$$b_2 b_{n-2} - b_n = (b_2^2 - b_4)\, b_{n-4}, \qquad n = 5, 6, 7, \ldots, \tag{4.3.2}$$

and

$$(b_1, b_2, b_3, b_4) \in \Omega_{10} \cup \Omega_{11}.$$

Consequently, the reciprocal coefficient region of S_5 *is* $\Omega_{10} \cup \Omega_{11}$.

Proof. Let $\psi(g) = z/g(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n) = z + a_3 z^3 + a_5 z^5 \in S_5$. For $n \ge 1$,

$$b_n = - \sum_{k=0}^{n-1} b_k\, a_{n-k+1}$$

where $a_2 = a_4 = a_k = 0$, for $k \ge 6$ and $b_o = 1$. Therefore,

$$b_1 = 0,\ b_2 = -a_3,$$

$$b_3 = 0,$$

and

$$b_4 = a_3^2 - a_5.$$

For $n \geq 5$,

$$b_n = -\sum_{k=2}^{n-1} a_k\, b_{n-k+1}$$

$$= -\,(-b_2\, b_{n-2} + (b_2^2 - b_4)\, b_{n-4})$$

$$= b_2\, b_{n-2} + (b_4 - b_2^2)\, b_{n-4}.$$

This gives the equation (4.3.2).

Now, we prove that the point $(b_1, b_2, b_3, b_4) \in \Omega_{10} \cup \Omega_{11}$. Using the condition (1.5.5) we have

(4.3.3) $$|b_2| \leq \frac{1+5(b_2^2-b_4)}{3}, \quad 0 \leq b_2^2-b_4 \leq 10^{-1}.$$

(4.3.4) $$|b_2| \leq 2\sqrt{(b_2^2-b_4)\,\{1-(b_2^2-b_4)\}} - (b_2^2-b_4),$$

$$10^{-1} \leq b_2^2-b_4 \leq 5^{-1}.$$

For $b_2 \geq 0$ the inequality (4.3.3) is equivalent to

$$b_4 - 11/100 \leq (b_2-3/10)^2 \qquad \text{when } 0 \leq b_2^2-b_4 \leq 10^{-1}.$$

This shows that $(b_1, b_2, b_3, b_4) \in \Omega_{10}$ since

$$(b_2-3/10)^2 \leq (b_2+3/10)^2.$$

For $b_2 \le 0$, the inequality (4.3.3) is equivalent to

$$b_4 - 11/100 \le (b_2 + 3/10)^2 \qquad \text{when} \quad 0 \le b_2^2 - b_4 \le 10^{-1}.$$

This implies that the point $(b_1, b_2, b_3, b_4) \in \Omega_{10}$ since

$$(b_2 + 3/10)^2 \le (b_2 - 3/10)^2.$$

Inequality (4.3.4) is equivalent to

$$3b_2^4 + 2b_2^3 - 3b_2^2 - 10b_2^2 b_4 - 2b_2b_4 + 5b_4^2 + 4b_4 \le 0,$$

$$10^{-1} \le b_2^2 - b_4 \le 5^{-1}$$

when $b_2 \ge 0$. This shows that the point $(b_1, b_2, b_3, b_4) \in \Omega_{11}$ since

$$3b_2^4 + 2b_2^3 - 3b_2^2 - 10b_2^2 b_4 - 2b_2b_4 + 5b_4^2 + 4b_4 \equiv A(b_2, b_4)$$

$$\ge A(-b_2, b_4).$$

The case when $b_2 \le 0$ follows similarly. Thus we have shown that the point $(b_1, b_2, b_3, b_4) \in \Omega_{10} \cup \Omega_{11}$.

Conversely, let $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ satisfy the equation (4.3.2) and the point $(b_1, b_2, b_3, b_4) \in \Omega_{10} \cup \Omega_{11}$. Set

$$\psi(g) = z/g(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

for $z \in U$. For $n \ge 1$,

$$a_{n+1} = -\sum_{k=1}^{n} b_k a_{n-k+1}.$$

Therefore,

$$a_2 = a_4 = a_6 = 0,\ a_3 = -b_2$$

and

$$a_5 = b_2^2 - b_4.$$

By the induction argument one can see that

$$a_{k+1} = 0 \text{ for } k \geq 5.$$

Therefore, for $z \in U$,

$$\psi(g) = z + a_3 z^3 + a_5 z^5.$$

Combining the condition (1.5.5) and that the point $(b_1, b_2, b_3, b_4) \in \Omega_{10} \cup \Omega_{11}$ gives that $\psi(g) \in S_5$. Hence the theorem is proved.

Examples. It follows from Theorem 4.3.2 that the functions $z/(1 + \sum_{n=1}^{\infty} z^{2n}/3^n)$ and $z/(1 + \sum_{n=2}^{\infty} z^{2n}/(-10)^{n-1})$ are in the class S_5.

Next we determine necessary and sufficient conditions in terms of the reciprocal coefficient region for a polynomial $\psi(g)$ (cf. (4.1.1)) to be in the class S_5^* (cf. Section 1.5).

Define

$$\Omega_{12} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3 = 0, \ (x_2-3/10)^2+0.11$$
$$\geq x_4 \leq (x_2+3/10)^2 + 0.11, \ -5^{-1} \leq x_2^2-x_4 \leq 10^{-1}\},$$

$$\Omega_{13} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3 = 0, \ 11x_2^2 -2x_2 - 26x_2x_4$$
$$+ 15x_4^2 + 3x_4 \leq 0, \ 10^{-1} \leq x_2-x_4 \leq 5^{-1}, \ x_2 \geq 0\}.$$

<u>Theorem</u> 4.3.3 <u>For the function</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>with</u> b_n <u>real</u>, $1 \leq n \leq 4$, $\psi(g)$ <u>is in the class</u> S_5^*, <u>if and only if, the equation</u> (4.3.2) <u>holds and</u>

<u>either</u> $(b_1, b_2, b_3, b_4) \in \Omega_{12}$ <u>or</u> $(b_1, b_2^2, b_3, b_4) \in \Omega_{13}$.

<u>Consequently, the reciprocal coefficient region of</u> S_5^* <u>is</u> $\Omega_{12} \cup \Omega_{13}$.

<u>Proof</u>. The proof is similar to that of Theorem 4.3.2 except that the inequality (1.5.3) has to be used in place of the inequality (1.5.5) and details of proof are omitted.

<u>Remark</u>. It can be observed that for $\psi(g) = z + a_3z^3 + a_5z^5 \in S_5^*$ with $a_5 \geq 0$, if the point $B = (b_1, b_2, b_3, b_4) \in \Omega_{12}$ in Theorem 4.3.3 where $\{b_n\}_{n=1}^{\infty}$ is the sequence of Taylor coefficients of $g(z)$, then the point B is also in the region Ω_{10} in Theorem 4.3.2

Examples. It follows from Theorem 4.3.3 that the functions $z/(1 + \sum_{n=2}^{\infty} z^{2n}/(\pm 5)^{n-1})$ are in the class S_5^*.

Finally we determine necessary and sufficient conditions in terms of the reciprocal coefficient region for a polynomial $\psi(g)$ to be in the class CV_5 (cf. Section 1.5).

Define,

$$\Omega_{14} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3 = 0, \ (x_2 + 9/50)^2 \geq x_4 - 19/50^2 \leq (x_2 - 9/50)^2, \ -1/25 \leq x_2^2 - x_4 \leq 1/50\},$$

$$\Omega_{15} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3 = 0, \ x_2 \geq 0, \ 1/50 \leq x_2 - x_4 \leq 1/25, \ 65x_2^2 - 190x_2x_4 + 125x_4^2 - 2x_2 + 5x_4 \leq 0\}.$$

Theorem 4.3.4 *For the function* $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *with* b_n *real*, $1 \leq n \leq 4$, $\psi(g)$ *is in the class* CV_5, *if and only if, the equation* (4.3.2) *holds and*

either $(b_1, b_2, b_3, b_4) \in \Omega_{14}$ *or* $(b_1, b_2^2, b_3, b_4) \in \Omega_{15}$.

Consequently, the reciprocal coefficient region of CV_5 *is* $\Omega_{14} \cup \Omega_{15}$

Proof. Alexander's type of relation between the classes CV_5, S_5^* and the method of proof of Theorem 4.3.3 enable us make the assertion of the theorem.

Remark. It can be verified that the regions Ω_{12} in Theorem 4.3.3 and Ω_{14} in Theorem 4.3.4 are related by

$$\Omega_{14} \subseteq \Omega_{12}.$$

Examples. It follows from Theorem 4.3.4 that the functions $z/(1 + \sum_{n=2}^{\infty} z^{2n}/(25)^{n-1})$, $z/(1 + \sum_{n=2}^{\infty} z^{2n}/(-50)^{n-1})$ are in the class CV_5.

CHAPTER V

COEFFICIENTS OF RECIPROCALS OF CERTAIN ANALYTIC FUNCTIONS

5.1 A function f in the class $\mathcal{A}_1$ of normalized (cf. Section 1.1) functions analytic in the unit disc U, with $f(z) \neq 0$ in the punctured disc $U\setminus\{0\}$, may be expressed as

$$(5.1.1) \qquad f(z) = \psi(g) = \frac{z}{g(z)} \quad \text{in } U,$$

where $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ in U. Such a function g is uniquely determined and we call it the reciprocal function of f in U. The largest (n-1)-dimensional region formed by the ordered (n-1) - tuples $(b_1, b_2, b_3, \ldots, b_{n-1})$, where $\psi(g)$ is in a certain class of univalent polynomials of degree atmost n $(n \geq 3)$ with univalent Gelfond-Leontev derivatives in U, is called the reciprocal coefficient region of that class of polynomials.

In this chapter we continue the study of properties of $\psi(g) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$ vis-a-vis the coefficients $\{b_n\}_{n=1}^{\infty}$ of

$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ when $\psi(g)$ is in certain subclasses of analytic functions.

Throughout in the sequel we denote, for $n \geq 2$, that

$$(5.1.2) \qquad T_{1,n}(D) = \{f \in T_1(D) : f(z) = z - a_2 z^2 - a_{n+1} z^{n+1}\},$$

$$(5.1.3) \qquad C_{1,n}(D) = \{f \in C_1(D) : f(z) = z - a_2 z^2 - a_{n+1} z^{n+1}\},$$

where, $T_1(D)$ and $C_1(D)$ are as in Definitions 1.4.13 and 1.4.14. For $d_n \equiv n$, $n \geq 1$, we denote $T_{1,n}(D) = T_{1,n}$ and $C_{1,n}(D) = C_{1,n}$.

We determine necessary and sufficient conditions in terms of the reciprocal coefficient regions for a trinomial $\psi(g)$ to be in the classes $T_{1,n}(D)$ or $C_{1,n}(D)$ respectively, $n \geq 2$, in Section 5.2.

In Section 5.3, we find the bounds of b_n, $1 \leq n \leq 4$, when a particular form of $\psi(g)$ is in $T_1(D)$ for a special sequence $\{d_n\}_{n=1}^{\infty}$ where, $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, $z \in U$. We determine in Section 5.4 sufficient conditions in terms of $\{b_n\}_{n=1}^{\infty}$ for the function $\psi(g)$ to be in the classes $S^*(A,B)$, $SP^{\lambda}(A,B)$, $P(A,B)$, $\mathcal{B}(\alpha)$ (cf. Section 1.3) or in a new class $ST(K)$. Some of these results generalize those of Reade et al. [109] and Ahuja and Jain [2]. Finally, in Section 5.5, we find necessary conditions in terms of $\{b_n\}_{n=1}^{\infty}$ when the function $\psi(g)$ is in one of the classes $C(\alpha)$, $ST(K)$, $S^*(\alpha)$, $SP(\lambda,\rho)$, $\mathcal{B}(\alpha)$ and $C_{\alpha}(K)$ (cf. Section 3.1) respectively.

5.2 This section is devoted to derive necessary and sufficient conditions in terms of the reciprocal coefficient regions for a trinomial $\psi(g)$ (cf. (5.1.1)) to be in the classes $T_{1,n}(D)$ and $C_{1,n}(D)$, $n \geq 2$ (cf. Section 5.1) respectively.

We begin with characterizing the reciprocal coefficient region of the class $T_{1,2}(D)$.

Define the region $\mathbb{D}_1$ by

$$\mathbb{D}_1 = \{(x,y) \in \mathbb{R}^2 : y-2/9 \leq (x-1/3)^2,\ x > 0 \text{ and }$$
$$x^2 \leq y \leq (x+d_2/(4d_3))^2 - (d_2/(4d_3))^2\}.$$

<u>Theorem</u> 5.2.1. <u>For the function</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>with</u> b_1, b_2 <u>real</u>, <u>the function</u> $\psi(g) = z-a_2z^2-a_3z^3$ $(a_2 > 0,\ a_3 \geq 0)$ <u>is in the class</u> $T_{1,2}(D)$, <u>if and only if</u>,

$$(5.2.1) \qquad b_1b_{n-1}-b_n = (b_1^2-b_2)b_{n-2} \qquad \text{for } n = 3,4,\ldots,$$

<u>and</u>

$$(b_1,b_2) \in \mathbb{D}_1.$$

<u>Consequently</u>, <u>the reciprocal coefficient region of</u> $T_{1,2}(D)$ <u>is</u> $\mathbb{D}_1$.

<u>Proof</u>. Let $\psi(g) = z-a_2z^2-a_3z^3 \in T_{1,2}(D)$ with $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$.

Now,

$$b_n = \sum_{k=0}^{n-1} b_k a_{n+1-k}$$

for $n \geq 1$ with $a_p = 0$ for $p \geq 4$. Hence,

$$b_1 = a_2, \text{ and } b_2 = a_3+b_1^2.$$

For $n \geq 3$,

$$b_n = a_2 b_{n-1} + a_3 b_{n-2} = b_1 b_{n-1} + (b_2-b_1^2) b_{n-2}.$$

This gives the equation (5.2.1).

Applying the inequality (1.5.1) for $\psi(g) = z-a_2z^2-a_3z^3$ in $T_{1,2}(D)$, we have

$$b_2-b_1^2 \leq \min \begin{cases} \dfrac{1-2b_1}{3} \\ \\ \dfrac{d_2 b_1}{2d_3} \end{cases}.$$

This implies,

$$b_2 - \frac{2}{9} \leq (b_1 - \frac{1}{3})^2$$

and

$$b_2 \leq (b_1 + \frac{d_2}{4d_3})^2 - (\frac{d_2}{4d_3})^2,$$

which in turn imply that $(b_1, b_2) \in \mathbb{D}_1$.

Conversely, let the equation (5.2.1) hold and $(b_1, b_2) \in \mathbb{D}_1$, for $\psi(g) = z - \sum_{n=2}^{\infty} a_n z^n$. Equation (5.2.1) gives that $\psi(g)$ is a cubic polynomial. Hence,

$$\psi(g) = z-a_2z^2-a_3z^3.$$

Since $(b_1, b_2) \in \mathbb{D}_1$, we have

$$a_2 = b_1 > 0, \ a_3 = b_2 - b_1^2 \geq 0$$

and

$$a_3 \leq \min\begin{cases} \dfrac{1-2a_2}{3} \\[2ex] \dfrac{d_2 a_2}{2d_3} \end{cases} .$$

Now the condition (1.5.1) gives that the function $\psi(g) \in T_{1,2}(D)$. The proof of the theorem is therefore complete.

Define (Figure 5.2.1),

$\mathbb{E}_1$ = the closure of the domain bounded by the parabolas $(x+1/3)^2 = y-2/9$, $(x-1/3)^2 = y - 2/9$ and $x^2 = y + 1/5$,

and

$$\mathbb{E}_2 = \mathbb{E}_1 \cap \{(x,y) \in \mathbb{R}^2 : x > 0, \ y > 0, \ x^2 \leq y \leq x^2+x/3\}.$$

Corollary 5.2.1. A function $\psi(g)$ is in the class $T_{1,2}$ with $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, ($b_1, b_2$ real), if and only if, the equation (5.2.1) holds and

$$(b_1, b_2) \in \mathbb{E}_2.$$

Consequently, the reciprocal coefficient region of $T_{1,2}$ is $\mathbb{E}_2$.

Proof. By choosing $\{d_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$ we have $\psi(g) = z-a_2z^2-a_3z^3$ is in $T_{1,2}(D)$, if and only if, $\psi(g) \in T_{1,2}$. Now the corollary follows from Theorem 5.2.1.

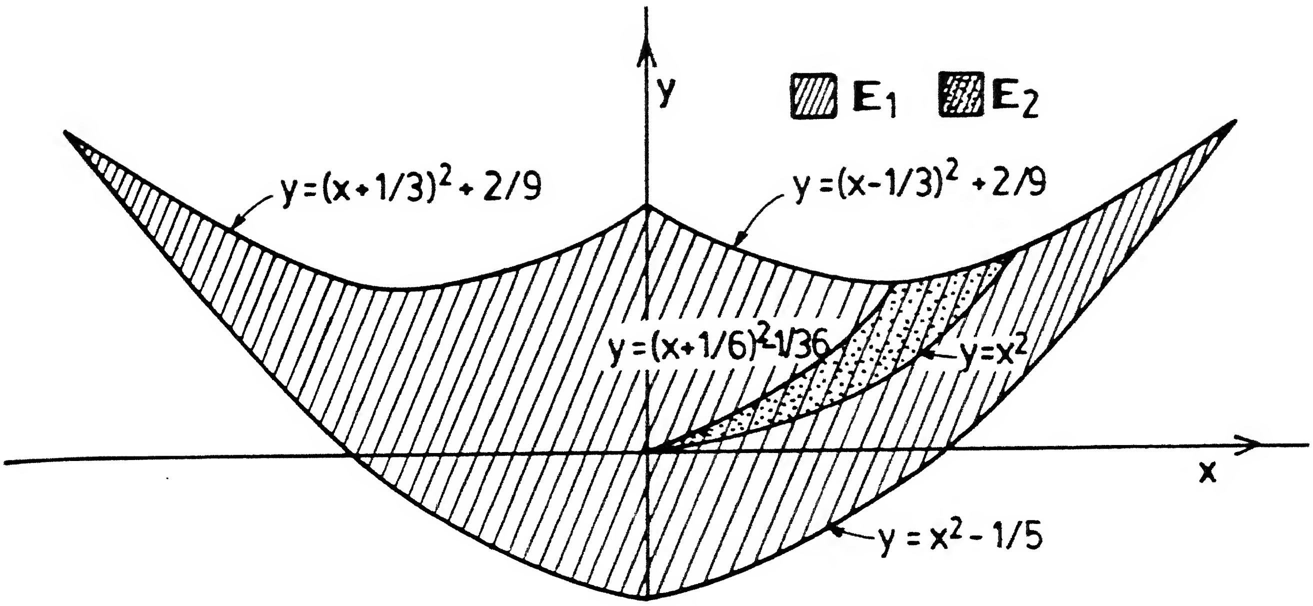

Fig. 5.2.1

Remark. It can be observed that the equations (1.5.6) and (5.2.1) are identical. Further the regions $\mathbb{E}_1$, $\mathbb{E}_2$ in Theorem 5.2.1 and the region D_1 determined by Silverman and Silvia [139] (cf. Section 1.5) are related by $\mathbb{E}_2 \subsetneq \mathbb{E}_1 = D_1$. Thus the reciprocal coefficient region of $T_{1,2}$ is an improvement over that of the class of normalized univalent cubic polynomials determined in [139].

Example. It follows from Corollary 5.2.1 that the function $z/(1+\sum_{n=1}^{\infty} z^n/2^n)$ is in the class $T_{1,2}$.

Next, we characterize the reciprocal coefficient region of the class $T_{1,p}(D)$, $p \geq 3$ (cf. (5.1.2)).

Define, for $p \geq 3$,

$$\mathbb{D}_2 = \Big\{(x_1,x_2,\ldots,x_p) \in \mathbb{R}^p : 0 < x_1,\ x_k = x_1^k \text{ for } 2 \leq k \leq p-1,$$
$$x_p - 1/(p+1) \leq x_1^p - 2x_1/(p+1) \text{ and}$$
$$x_1^p \leq x_p \leq x_1(x_1^{p-1} + d_2/(pd_{p+1}))\Big\}.$$

Theorem 5.2.2. *For the function* $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *with* b_n *real for* $1 \leq n \leq p$, $p \geq 3$, *the function* $\psi(g) = z-a_2z^2-a_{p+1}z^{p+1}$ *is in* $T_{1,p}(D)$, *if and only if*,

$$b_1b_{n-1} - b_n = (b_1^p-b_p)b_{n-p} \qquad \text{for } n \geq p+1, \tag{5.2.2}$$

and

$$(b_1, b_2, \ldots, b_p) \in \mathbb{D}_2.$$

Consequently, the reciprocal coefficient region of $T_{1,p}(D)$ is $\mathbb{D}_2$.

Proof. Let $\psi(g) = z - \sum_{n=2}^{\infty} a_n z^n \in T_{1,p}(D)$ with $a_k = a_s = 0$ for $3 \le k \le p$, $s \ge p+2$. Now, for $n \ge 1$,

$$b_n = \sum_{k=0}^{n-1} b_k a_{n+1-k}.$$

Hence,

$$b_1 = a_2 \text{ and } b_2 = b_1^2.$$

Assume that $b_m = b_1^m$ for $1 \le m \le q$ where $q \in \mathbb{N}$ and $q \le p-2$. Now,

$$b_{q+1} = b_1^{q+1}.$$

So,

$$b_i = b_1^i \text{ for } 1 \le i \le p-1 \text{ and } b_p = b_1^p + a_{p+1}.$$

For $n \ge p+1$, we have

$$b_n = b_1 b_{n-1} + a_{p+1} b_{n-p}.$$

This gives the equation (5.2.2).

Applying the inequality (1.5.1) for $\psi(g) = z-a_2z^2-a_{p+1}z^{p+1}$ in $T_1(D)$, we have

$$b_p - b_1^p \le \min \begin{cases} \dfrac{1-2b_1}{p+1} \\ \\ \dfrac{d_2 b_1}{p d_{p+1}} \end{cases}$$

which implies

$$b_p - \frac{1}{p+1} \le b_1^p - \frac{2b_1}{p+1}$$

and

$$b_p \le b_1 (b_1^{p-1} + \frac{d_2}{p d_{p+1}}) .$$

Thus, we have $(b_1, b_2, \ldots, b_p) \in \mathbb{D}_2$.

Conversely, let the equation (5.2.2) hold and the point $(b_1, b_2, \ldots, b_p) \in \mathbb{D}_2$, for

$$\psi(g) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad z \in U.$$

For $n \ge 1$, we have

$$a_{n+1} = b_n - \sum_{k=2}^{n} a_k b_{n-k+1}.$$

Hence

$$a_2 = b_1 \text{ and } a_3 = 0.$$

Let $p > 3$. Assume that $a_s = 0$ for $3 \le s \le j$ where, $3 \le j \le p-1$ and $j \in \mathbb{N}$. Now,

$$a_{j+1} = b_j - b_1^j = 0.$$

Therefore,

$$a_m = 0 \text{ for } 4 \le m \le p.$$

Now

$$a_{p+1} = b_p - b_1^p.$$

It is easily seen that the last equation is true for $p = 3$ also. Making use of the equation (5.2.2) and induction, we get that

$$\psi(g) = z - a_2 z^2 - a_{p+1} z^{p+1}.$$

The point $(b_1, b_2, \ldots, b_p) \in \mathbb{D}_2$ and the condition (1.5.1) give that $\psi(g) \in T_1(D)$. Hence the proof of the theorem is complete.

Next, we characterize the reciprocal coefficient region of the class $C_{1,2}(D)$ (cf. (5.1.3)).

Define

$$\mathbb{D}_3 = \left\{ (x,y) \in \mathbb{R}^2 : x^2 \le y \le (x - \frac{2}{9})^2 + \frac{5}{81},\ x > 0 \text{ and } y + (\frac{d_2}{8d_3})^2 \le (x + \frac{d_2}{8d_3})^2 \right\}.$$

<u>Theorem</u> 5.2.3 <u>For the function</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>with</u> b_1, b_2 <u>real</u>, $\psi(g) = z - a_2 z^2 - a_3 z^3 \in C_{1,2}(D)$, <u>if and only if, the equation</u> (5.2.1) <u>holds and</u>

$$(b_1, b_2) \in \mathbb{D}_3.$$

<u>Consequently, the reciprocal coefficient region of the class $C_{1,2}(D)$ is $\mathbb{D}_3$.</u>

<u>Proof</u>. Follows on the lines of the proof of Theorem 5.2.1 with the inequality (1.5.2) replacing the inequality (1.5.1).

<u>Remark</u>. It can be observed that the regions $\mathbb{D}_1$ in Theorem 5.2.1 and $\mathbb{D}_3$ in Theorem 5.2.3 are related by $\mathbb{D}_3 \subseteq \mathbb{D}_1$.

Define (Figure 5.2.2),

$$\mathbb{E}_3 = \left\{(x,y) \in \mathbb{R}^2 : x > 0,\ x^2 - 5/9^2 \le y - 5/9^2 \le (x-2/9)^2 \right.$$
$$\left. \text{and } 0 < y \le x^2 + x/6\right\}.$$

<u>Corollary</u> 5.2.2. An analytic function $\psi(g)$ is in $C_{1,2}$ with $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, ($b_1, b_2$ real), if and only if, the equation (5.2.1) holds and

$$(b_1, b_2) \in \mathbb{E}_3.$$

Consequently, the reciprocal coefficient region of $C_{1,2}$ is $\mathbb{E}_3$.

<u>Proof</u>. By choosing $\{d_n\}_{n=1}^{\infty} \equiv \{n\}_{n=1}^{\infty}$ we have $\psi(g) = z-a_2z^2-a_3z^3$ is in $C_{1,2}(D)$, if and only if, $\psi(g) \in C_{1,2}$. Now the corollary follows from Theorem 5.2.3.

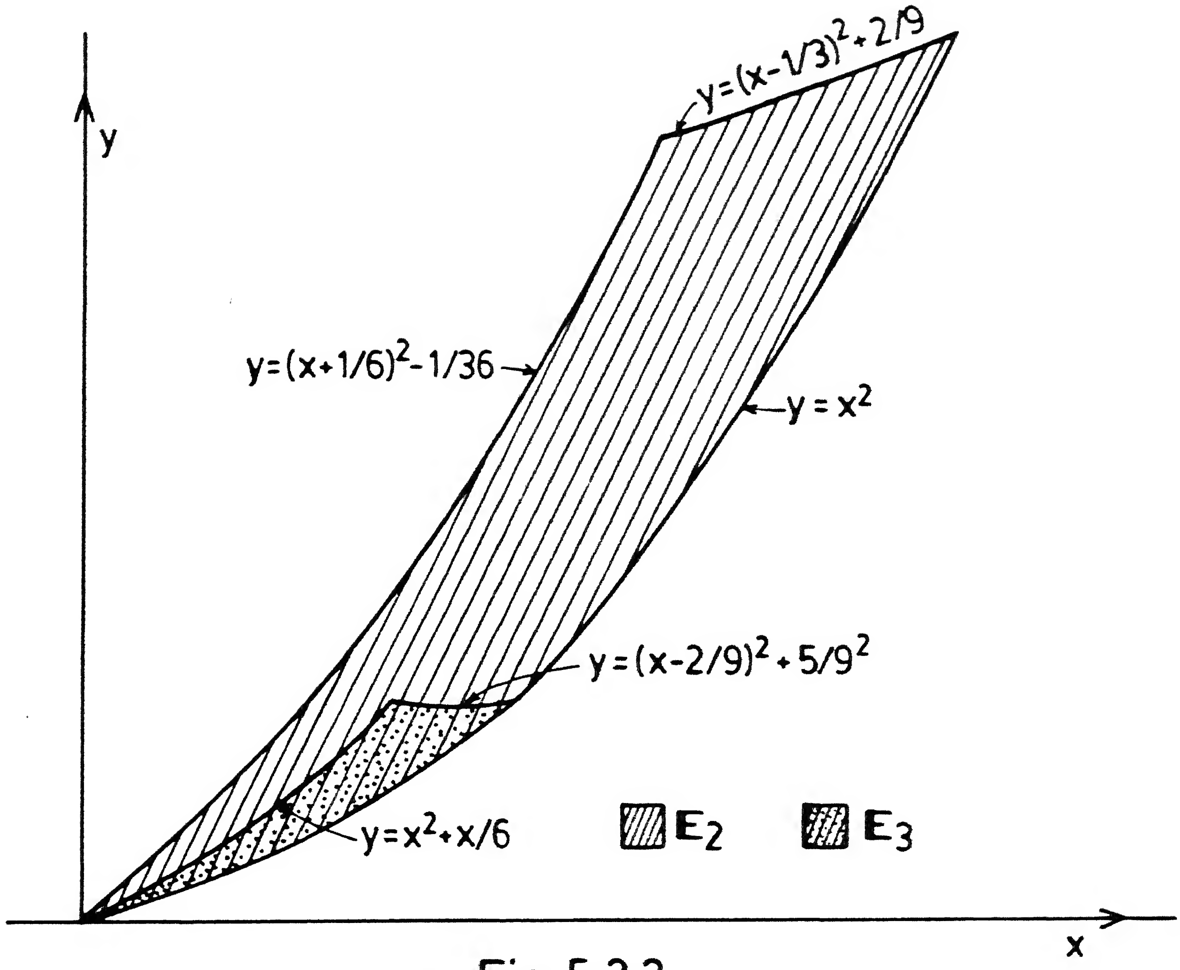

Fig. 5.2.2

<u>Remark</u>. It can be verified that the regions $\mathbb{E}_2$ in Corollary 5.2.1 and $\mathbb{E}_3$ in Corollary 5.2.2 are related by

$$\mathbb{E}_3 \subsetneq \mathbb{E}_2.$$

<u>Example</u>. It follows from Corollary 5.2.2 that the function $z/(1 + \sum_{n=1}^{\infty} z^n/4^n)$ is in the class $C_{1,2}$.

Next, we characterize the reciprocal coefficient region of the class $C_{1,p}(D)$, $p \geq 3$ (cf. (5.1.3)).

Define, for $p \geq 3$,

$$\mathbb{D}_4 = \Big\{(x_1,x_2,\ldots,x_p) \in \mathbb{R}^p : 0 < x_1,\ x_k = x_1^k \text{ for } 2 \leq k \leq p-1,$$
$$x_p - (p+1)^{-2} \leq x_1(x_1^{p-1}-4(p+1)^{-2})$$
$$\text{and } x_1^p \leq x_p \leq x_1(x_1^{p-1} + d_2/(p^2 d_{p+1}))\Big\}.$$

<u>Theorem 5.2.4</u>. <u>For the function</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>with</u> b_k <u>real for</u> $1 \leq k \leq p$, $p \geq 3$, <u>the function</u> $\psi(g) = z-a_2z^2-a_{p+1}z^{p+1}$ is in $C_{1,p}(D)$, <u>if and only if, the equation</u> (5.2.2) <u>holds and</u>

$$(b_1,b_2,\ldots,b_p) \in \mathbb{D}_4.$$

<u>Consequently, the reciprocal coefficient region of</u> $C_{1,p}(D)$ <u>is</u> $\mathbb{D}_4$.

<u>Proof</u>. Follows by using the method of Proof of Theorem 5.2.2 with

the inequality (1.5.2) replacing the inequality (1.5.1).

Remark. It can be verified that the regions $\mathbb{D}_2$ in Theorem 5.2.2 and $\mathbb{D}_4$ in Theorem 5.2.4 are related by

$$\mathbb{D}_4 \subseteq \mathbb{D}_2.$$

5.3 In this section we find the bounds of b_n's, $1 \leq n \leq 4$, when a particular form of $\psi(g)$ (cf. (5.1.1)) is in the class $T_1(D)$ (cf. Definition 1.4.13), for a special sequence $\{d_n\}_{n=1}^{\infty}$ where $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ in U.

It is known [130] that for $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$ (cf. (1.4.2)), $a_2 \leq 1/2$. Thus if, $\psi(g) \in T_1(D)$ with $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, as its reciprocal in U, then it follows that

$$0 < b_1 \leq 1/2, \tag{5.3.1}$$

since $T_1(D) \subseteq T$. The function

$$t(z) = z\left(1 + \sum_{n=1}^{\infty} (2^{-1}z)^n\right)^{-1} \tag{5.3.2}$$

gives sharpness in the right hand side inequality of (5.3.1). The left hand side inequality of (5.3.1) is sharp in the sense that for every $b_1 > 0$, there exists a function

$$\psi(g) = z/g(z) = z/\left(1 + \sum_{n=1}^{\infty} b_n z^n\right) \in T_1(D).$$

This can be seen with the help of the functions

$$h(z) = z\left(1 + \sum_{n=1}^{\infty} (b_1 z)^n\right)^{-1} = z - b_1 z^2, \quad 0 < b_1 < 1/2 \tag{5.3.3}$$

in $T_1(D)$.

Next we consider cubic polynomials $\psi(g)$ in $T_1(D)$ for a special sequence $\{d_n\}$ and find the bound on the second Taylor series coefficient b_2 of $g(z)$.

<u>Theorem</u> 5.3.1. <u>For the function</u> $\psi(g) = z - a_2 z^2 - a_3 z^3 \in T_1(D)$, <u>where</u> $3d_2 \le 2d_3$ <u>and</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>in</u> U, <u>we have</u>

$$0 < b_2 \le 4^{-1}. \tag{5.3.4}$$

<u>The sharpness in the upper inequality of</u> (5.3.4) <u>is attained for the function</u> t(z) <u>given by</u> (5.3.2).

<u>Proof</u>. By the inequality (1.5.1) we have,

$$b_2 - b_1^2 \le \min \begin{cases} \dfrac{1 - 2b_1}{3} \\[2ex] \dfrac{d_2 b_1}{2d_3} \end{cases}.$$

When $0 < b_1 \le 2d_3/(3d_2 + 4d_3)$, we have

$$\frac{d_2 b_1}{2d_3} \le \frac{1-2b_1}{3}.$$

Consequently,

$$b_2 - b_1^2 \le \frac{d_2 b_1}{2d_3} .$$

$$b_2 \le b_1^2 + \frac{d_2 b_1}{2d_3} < 1/4.$$

When $2d_3/(3d_2 + 4d_3) \le b_1 \le 1/2$, we have

$$\frac{1-2b_1}{3} \le \frac{d_2 b_1}{2d_3} .$$

Therefore, we obtain that

$$b_2 - b_1^2 \le \frac{1 - 2b_1}{3} ,$$

$$b_2 \le b_1^2 - \frac{2b_1}{3} + \frac{1}{3} \le \frac{1}{4} .$$

Hence the theorem is proved. The lower inequality in (5.3.4) is sharp in the sense that for every $y > 0$, there exists a function $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, $b_2 = y$, such that $\psi(g) \in T_1(D)$ and is a cubic polynomial. This can be observed with the help of the functions $h(z)$ given in (5.3.3).

Next we consider a trinomial $\psi(g)$ in $T_1(D)$ for a special sequence $\{d_n\}$ and find the bound on the third Taylor series coefficient, b_3 of $g(z)$.

<u>Theorem</u> 5.3.2. <u>For the function</u> $\psi(g) = z-a_2z^2-a_4z^4 \in T_1(D)$ <u>where</u> $4d_2 \le 3(\sqrt{6}-2)d_4$ <u>and</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_nz^n$ <u>in</u> U, <u>we have</u>,

(5.3.5) $$0 < b_3 \le 8^{-1}.$$

<u>The function</u> t(z) <u>defined in</u> (5.3.2) <u>gives sharpness in the upper inequality of</u> (5.3.5).

<u>Proof</u>. By the inequality (1.5.1), we have

$$b_3 - b_1^3 \le \min \begin{cases} \dfrac{1-2b_1}{4} \\ \\ \dfrac{d_2b_1}{3d_4} \end{cases}$$

When $0 < b_1 \le 3d_4/(2(2d_2 + 3d_4))$, we have

$$\frac{d_2b_1}{3d_4} \le \frac{1}{4}(1-2b_1).$$

Consequently,

$$b_3-b_1^3 \le \frac{d_2b_1}{3d_4} .$$

$$b_3 \le b_1^3 + \frac{d_2b_1}{3d_4} < \frac{1}{8}.$$

When $3d_4/(2(2d_2 + 3d_4)) < b_1 \le 1/2$, we have

$$\frac{1}{4}(1-2b_1) < \frac{d_2b_1}{3d_4}.$$

Hence, we obtain that

$$b_3 - b_1^3 \le \frac{1}{4}(1-2b_1),$$

$$b_3 \le b_1^3 - \frac{b_1}{2} + \frac{1}{4} \le \frac{1}{8}.$$

Thus the proof of the theorem is complete. The lower bound is sharp in the sense that for every $\epsilon > 0$, there exists a function $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, $b_3 = \epsilon$, and $\psi(g) = z-a_2z^2-a_4z^4 \in T_1(D)$. This can be seen with the help of the functions $h(z)$ defined in (5.3.3).

Next we consider a trinomial $\psi(g)$ in T_1 and find the bound on the fourth Taylor series coefficient, b_4 of $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ which gives that, for trinomial $\psi(g) \in T_1$, the bound $b_n \le 1/2^n$ may not be true in general for $n \ge 1$.

<u>Theorem</u> 5.3.3. <u>For the function</u> $\psi(g) = z-a_2z^2-a_5z^5 \in T_1$ <u>and</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>in</u> U,

$$0 < b_4 \le \frac{41}{625}. \tag{5.3.6}$$

<u>The function</u> $\psi(g) = z - (2/5)z^2 - z^5/25 = z/(1 + (2/5)z + (4/25)z^2 + (8/125)z^3 + (41/625)z^4 + \ldots)$ <u>gives shaprness in the upper inequality of</u> (5.3.6).

<u>Proof</u>. By the inequality (1.5.1), we have

$$b_4 - b_1^4 \le \min \begin{cases} \dfrac{1 - 2b_1}{5} \\ \\ \dfrac{b_1}{10} . \end{cases}$$

When $0 < b_1 \le 2/5$, we have

$$\frac{b_1}{10} \le \frac{1-2b_1}{5} ,$$

$$b_4 - b_1^4 \le \frac{b_1}{10} ,$$

$$b_4 \le b_1^4 + \frac{b_1}{10} \le \frac{41}{625} .$$

When $2/5 \le b_1 \le 1/2$, we have

$$\frac{(1 - 2b_1)}{5} \le \frac{b_1}{10} .$$

Consequently,

$$b_4 \le b_1^4 - \frac{2b_1}{5} + \frac{1}{5} \le \frac{41}{625} .$$

Hence the proof of the theorem is complete. With the help of the functions $h(z)$ defined in (5.3.3) it can be seen that the lower bound of b_4 in (5.3.6) is sharp in the sense that for every $\epsilon > 0$, there exists a function $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, $b_4 = \epsilon$, and $\psi(g)$ is in T_1.

Finally, we find a bound on the second Taylor series coefficient b_2 of $g(z)$ when $\psi(g) \in T_1$ is not necessarily a polynomial.

<u>Theorem</u> 5.3.4. <u>Let</u> $\psi(g) \in T_1(D)$ <u>and</u> $\beta_o \equiv \sup_{n\geq 2} [d_n/n]$. <u>If</u> $\beta_o < \infty$, <u>and</u> $0 < b_1 \leq \beta_o/d_2$, <u>then</u>

$$(5.3.7) \qquad 0 < b_2 < \frac{2\beta_o+d_3}{4d_3}.$$

<u>Set</u> $\tilde{\beta} = \sup\{b_2 : \psi(g) \in T_1(D)\}$. <u>If in addition to the above conditions</u>, $2d_3 \geq 3d_2$, <u>then</u>

$$(5.3.8) \qquad \frac{1}{4} \leq \tilde{\beta} \leq \frac{2\beta_o + d_3}{4d_3}.$$

<u>Proof</u>. We have $b_2 = a_3 + b_1^2$ where $\psi(g) = z - \sum_{n=2}^{\infty} a_n z^n$ in U. This and the inequality (1.4.18) together give that

$$b_2 < \frac{\beta_o}{2d_3} + \frac{1}{4}$$

by using the inequality (5.3.1). This gives the inequality (5.3.7).

By using Theorem 5.3.1, we obtain the inequality (5.3.8) and this completes proof of the theorem.

Corollary 5.3.1. If $\psi(g) \in T_1$ then $0 < b_2 < 5/12$. Set, $\beta = \sup \{b_2 : \psi(g) \in T_1\}$. Then, $1/4 \le \beta \le 5/12$.

Proof. The corollary follows from Theorem 5.3.5 by choosing $\{d_n\}_{n=1}^{\infty}$ to be the sequence $\{n\}_{n=1}^{\infty}$.

5.4 We determine sufficient conditions in terms of the Taylor series coefficients $\{b_n\}_{n=1}^{\infty}$ of $g(z)$ for the function $\psi(g)$ (cf.(5.1.1)) to be in the classes $S^*(A,B)$, $SP^{\lambda}(A,B)$, $P(A,B)$, $\mathcal{B}(\alpha)$ (cf. Section 1.3) or in a new class ST(K) defined in this section.

Define,

$$ST(K) \equiv \{zf' : f \in C(K)\},$$

where $1 \ge K > 0$, and $C(K)$ is as in Section 1.2. It is easily seen that the class $ST(R_1,R_2) \subseteq ST(1/R_2)$ for $1 \le R_2 < \infty$, where $ST(R_1,R_2)$ is as in Section 1.2.

We begin with deriving a sufficient condition for the class ST(K).

Theorem 5.4.1. Let $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n)$ in U, b_n's are complex and

$$\sum_{n=1}^{\infty} (n+1)|b_n| \le 1-K$$

where $0 < K < 1$. Then, $f \in ST(K)$.

Proof. By putting $f(z) = z/g(z)$ in U, we have

$$\mathrm{Re}\, \frac{zf'(z)}{|f(z)|f(z)} = \left|\frac{g(z)}{z}\right| \mathrm{Re}\left(1 - \frac{zg'(z)}{g(z)}\right)$$

$$\geq \left|\frac{g(z)}{z}\right|\left(1 - \left|\frac{zg'(z)}{g(z)}\right|\right)$$

$$= \frac{1}{|z|}\left(|g(z)| - |zg'(z)|\right)$$

$$\geq \frac{1}{|z|}\left(1 - \sum_{n=1}^{\infty} |b_n| - \sum_{n=1}^{\infty} n|b_n|\right)$$

$$= \frac{1}{|z|}\left(1 - \sum_{n=1}^{\infty} (n+1)|b_n|\right)$$

$$\geq \frac{K}{|z|} > K$$

in $U\backslash\{o\}$. Thus the theorem is proved.

Next a sufficient condition is determined for the class $SP^{\lambda}(A,B)$ (cf. Section 1.3).

Thereom 5.4.2. Let $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n)$, $z \in U$, b_n's be complex, $-1 \leq B < A \leq 1$, and λ be real with $|\lambda| < \pi/2$. If the coefficients b_n's satisfy

$$(5.4.1) \qquad \sum_{n=1}^{\infty} (n + |(B-A)\cos\lambda - n B e^{i\lambda}|)\, |b_n| \leq (A-B)\cos\lambda$$

then, f is in the class $SP^{\lambda}(A,B)$.

Proof. By setting,

$$\frac{e^{i\lambda}\frac{zf'(z)}{f(z)} - i\sin\lambda}{\cos\lambda} = \frac{1+Aw(z)}{1+Bw(z)}$$

we have

$$w(z) = \frac{e^{i\lambda}\sum_{n=1}^{\infty} nb_n z^n}{(B-A)\cos\lambda + \sum_{n=1}^{\infty}\{(B-A)\cos\lambda - Bne^{i\lambda}\} b_n z^n},$$

$$|w(z)| \le \frac{\sum_{n=1}^{\infty} n|b_n|}{(A-B)\cos\lambda - \sum_{n=1}^{\infty}|(B-A)\cos\lambda - nBe^{i\lambda}||b_n|}$$

$$\le 1$$

in U, by the condition (5.4.1). Thus, $w(z)$ is analytic in U, $w(0) = 0$ and $|w(z)| \le 1$. Now, in view of the mapping properties of the function $(1+Az)/(1+Bz)$, (cf. Section 1.3), the theorem is proved.

Taking $A = (1-2\rho)$ and $B = -1$, $0 \le \rho < 1$ in Theorem 5.4.2, the following sufficient condition for the class $SP(\lambda,\rho)$ (cf. Section 1.3) is obtained.

<u>Corollary</u> 5.4.1 (Ahuja and Jain [2]). If $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n)$ in U, b_n's are complex, λ is real with $|\lambda| < \pi/2$ and

$$\sum_{n=1}^{\infty} (n + |ne^{i\lambda} - 2(1-\rho) \cos \lambda|) |b_n| \le 2(1-\rho) \cos \lambda$$

then $f \in SP(\lambda,\rho)$.

Taking $\lambda = 0$ in Theorem 5.4.2, the following sufficient condition for the class $S^*(A,B)$ (cf. Section 1.3) is obtained.

<u>Corollary</u> 5.4.2. If $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n)$, $z \in U$, b_n's are complex, $-1 \le B < A \le 1$, and the coefficients b_n's satisfy

$$\sum_{n=1}^{\infty} (n+|B(1-n)-A|) |b_n| \le A-B$$

then, $f \in S^*(A,B)$.

<u>Remark</u>. For $A = 1 - 2\alpha$, $B = -1$, Corollary 5.4.3 gives Theorem 1 of Reade et al. [109].

Next a sufficient condition is determined for the class P(A,B) (cf. Section 1.3).

<u>Theorem</u> 5.4.3. <u>If</u> $f(z) = 1/(1 + \sum_{n=1}^{\infty} b_n z^n)$, <u>for</u> $z \in U$, b_n's <u>are complex</u>, <u>and</u>

$$\sum_{n=1}^{\infty} (1 + |A|) |b_n| \le A-B \tag{5.4.2}$$

<u>for</u> $-1 \le B < A \le 1$, <u>then</u> $f \in P(A,B)$.

<u>Proof</u>. By setting,

$$f(z) = \frac{1+Aw(z)}{1+Bw(z)}$$

we have

$$w(z) = \frac{\sum_{n=1}^{\infty} b_n z^n}{B-A-A\sum_{n=1}^{\infty} b_n z^n} .$$

Further,

$$|w(z)| \le \frac{\sum_{n=1}^{\infty} |b_n|}{A-B - |A| \sum_{n=1}^{\infty} |b_n|} \le 1$$

in U, by the coefficient condition (5.4.2). Thus, $w(z)$ is analytic, $|w(z)| \le 1$ in U and $w(0) = 0$. Hence, $f \in P(A,B)$ follows.

<u>Corollary</u> 5.4.3. <u>For</u> $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n)$, $z \in U$, b_n's complex,

(i) <u>if</u> $\sum_{n=1}^{\infty} |b_n| \le 1$, <u>then</u> $f \in \mathcal{S}(\alpha)$, $0 \le \alpha \le 1/2$,

(ii) <u>if</u> $\sum_{n=1}^{\infty} |b_n| \le \frac{1-\alpha}{\alpha}$, <u>then</u> $f \in \mathcal{S}(\alpha)$, $1/2 \le \alpha < 1$.

<u>Proof</u>. The corollary follows from Theorem 5.4.3 and the observation that for $A = 1-2\alpha$, $B = -1$ we have $f \in \mathcal{S}(\alpha)$, if and

only if, $f/z \in P(A,B)$.

5.5 In this section necessary conditions are determined in terms of the Taylor series coefficients b_n's of $g(z)$ when the function $\psi(g)$ (cf. (5.1.1)) is in one of the classes $ST(K)$ (cf. Section 5.4), $S^*(\alpha)$, $SP(\lambda,\rho)$, $\mathcal{B}(\alpha)$ and $C_\alpha(K)$ (cf. Section 3.1).

We begin with proving a necessary condition for functions in the class $ST(K)$ of special form.

Theorem 5.5.1 *If* $\psi(g) = \dfrac{z}{1 + \sum_{n=1}^{\infty} |b_n| z^n} \in ST(K)$, b_n's *are complex and* $0 < K \le 1$, *then*

$$\sum_{n=2}^{\infty} (n-1)\,|b_n| \le 1-K. \tag{5.5.1}$$

Proof. For $f(z) = \psi(g) = z/g(z) = z/(1 + \sum_{n=1}^{\infty} |b_n| z^n)$, $z \in U$, we have

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)\;|f(z)|} = \mathrm{Re}\,\frac{(g(z) - zg'(z))\;|g(z)|}{g(z)\;|z|}$$

$$> K$$

in $U\backslash\{0\}$, by the Alexander type relation between the classes $C(K)$, $ST(K)$ and the local minimum property (1.2.54) for the curvature, $k(f;z)$. Thus, for $z \in (0,1)$, we have

$$\mathrm{Re}\left[1 + \sum_{n=2}^{\infty} (1-n)\ |b_n|z^n\right] > Kz.$$

By letting z tend to 1^- along the positive reals, we obtain the inequality (5.5.1).

<u>Remark</u>. For $\psi(g) = z/(1 + \sum_{n=1}^{\infty} |b_n|z^n) \in ST(K)$, the inequality (5.5.1) is stronger than the inequality (1.6.2) of Prawitz [103], where $0 < K < 1$ and b_n's are complex.

Next a necessary condition is derived for functions of particular form in the class $SP(\lambda,\rho)$ (cf. Section 1.3).

<u>Theorem</u> 5.5.2. <u>If</u> $\psi(g) = z/(1 + \sum_{n=1}^{\infty} |b_n|z^n) \in SP(\lambda,\rho)$, b_n's <u>are complex</u> λ <u>is real with</u> $|\lambda| < \pi/2$, $0 \le \rho < 1$, <u>then</u>

$$(5.5.2) \qquad \sum_{n=1}^{\infty} (n-1+\rho)\ |b_n| \le 1-\rho.$$

<u>Proof</u>. For $f(z) = \psi(g) = z/(1 + \sum_{n=1}^{\infty} |b_n|z^n) \in SP(\lambda,\rho)$, we have,

$$\mathrm{Re}\ \frac{e^{i\lambda}\ zf'(z)}{f(z)} = \mathrm{Re}\ \frac{e^{i\lambda}(\sum_{n=2}^{\infty} (1-n)|b_n|z^n)}{1 + \sum_{n=1}^{\infty} |b_n|z^n}$$

$$> \rho \cos \lambda$$

in U. Now letting z tend to 1^- along positive reals, the inequality (5.5.2) is obtained.

Remark. The inequality (5.5.2) is stronger than the inequality (1.6.2) of Prawitz [103] for functions $\psi(g) = z/(1 + \sum_{n=1}^{\infty} |b_n|z^n)$ in $SP(\lambda,\rho)$, when b_n's are complex.

Taking $\lambda = 0$ and $\rho = \alpha$, in Theorem 5.5.2 the next result is obtained.

Corollary 5.5.1. If $\psi(g) = z/(1 + \sum_{n=1}^{\infty} |b_n|z^n) \in S^*(\alpha)$, $0 \le \alpha < 1$, and b_n's are complex, then

$$(5.5.3) \qquad \sum_{n=1}^{\infty} (n-1+\alpha)\, |b_n| \le 1-\alpha.$$

Remarks. (i) Necessary condition (5.5.3) and sufficient condition (1.6.4) suggest that an analytic function $\psi(g) = z/(1+ \sum_{n=1}^{\infty} |b_n|z^n)$, $z \in U$, b_n's are complex, is in the class $S^*(\alpha)$, $1/2 \le \alpha < 1$, if and only if, the inequality (5.5.3) holds.

(ii) It can be verified, for functions $\psi(g)=z/(1 + \sum_{n=1}^{\infty} |b_n|z^n)$ in $\mathcal{B}(\alpha)$ (cf. Section 1.3), $1/2 \le \alpha < 1$, b_n's are complex, that

$$(5.5.4) \qquad \sum_{n=1}^{\infty} |b_n| \le \frac{1-\alpha}{\alpha} .$$

Thus, in view of (ii) of Corollary 5.4.3, a necessary and sufficient condition for the function $\psi(g) = z/(1+ \sum_{n=1}^{\infty} |b_n|z^n)$,

b_n's are complex, to be in $\mathcal{B}(\alpha)$, $1/2 \le \alpha < 1$, is that the condition (5.5.4) holds.

Next we determine a necessary condition on the first two Taylor series coefficients b_1 and b_2 of $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ when the function $\psi(g) \in C_\alpha(K)$ (cf. Section 3.1).

<u>Theorem</u> 5.5.3. <u>For</u> $\psi(g) = z/(1 + \sum_{n=1}^{\infty} b_n z^n) \in C_\alpha(K)$, $0 < K < 1$, $\alpha < 1$, <u>we</u> <u>have</u>

$$|b_1|^2 \le (1-\alpha)^2 \, (1-K) \tag{5.5.5}$$

<u>and</u>

$$\frac{3}{1-\alpha} \, |b_2| \le \frac{1 - K - |b_1|^2 \{1-\alpha(1-K/2)\}(1-\alpha)^{-2}}{1-K/2} . \tag{5.5.6}$$

<u>Both</u> <u>the</u> <u>inequalities</u> <u>are</u> <u>sharp</u>.

<u>Proof</u>. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n = \psi(g)$, we have,

$$b_1 = -a_2 \text{ and } b_2 = a_2^2 - a_3.$$

By Theorem 3.3.3, we have that

$$|a_2|^2 \le (1-\alpha)^2 \, (1-K).$$

By substituting $b_1 = -a_2$ in this, we obtain the inequality (5.5.5).

By (3.3.9), we have

$$\frac{3}{1-\alpha}\left|a_3 - \frac{1-2\alpha/3}{1-\alpha}a_2^2\right| \le \frac{1 - |a_2|^2(1-\alpha)^{-2} - K}{1-K/2} .$$

Hence,

$$\frac{3}{(1-\alpha)}|a_3 - a_2^2| - \frac{\alpha}{(1-\alpha)^2}|a_2|^2 \le \frac{1 - |a_2|^2(1-\alpha)^{-2} - K}{1-K/2} ,$$

$$\frac{3}{1-\alpha}|a_3 - a_2^2| \le \frac{1-K - |a_2|^2(1-\alpha(1-K/2))(1-\alpha)^{-2}}{1-K/2} .$$

Now by substituting b_2 for $a_2^2 - a_3$ and b_1 for $-a_2$ in the last inequality we obtain the inequality (5.5.6). The functions $((1+az)^{2\alpha-1}-1)/((2\alpha-1)a))$ for $\alpha \ne 1/2$; $e^{i\phi}a^{-1}\log(1+\bar{a}z) + b$ for $\alpha = 1/2$ and $e^{i\phi}(1-|a|^2)^{\alpha-1}z + b$, with $K = 1-|a|^2$, $a \in U\backslash\{0\}$, $b \in \mathbb{C}$ and $\phi \in \mathbb{R}$ give sharpness in the inequalities (5.5.5) and (5.5.6). This completes the proof of the theorem.

Finally in this section, we determine a necessary condition in terms of the Taylor series coefficients of $g(z)$ when $\psi(g)$ (cf. (5.1.1)) is in the class $C(\alpha)$ (cf. Section 1.4). Set, $b_0 = 1$.

<u>Theorem</u> 5.5.4. <u>If</u> $\psi(g) \in C(\alpha)$, $0 \le \alpha < 1$ <u>with</u> $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ <u>in</u> U, <u>then</u>

$$(5.5.7) \qquad |b_n| \le \frac{1-\alpha}{(n+1)(n+1-\alpha)} , \qquad n = 0,1,2,\ldots .$$

<u>The inequality is sharp for</u> $g_n(z) = 1+\sum_{k=1}^{\infty}[(1-\alpha)/(n+1)(n+1-\alpha)]^k z^{nk}$

and $\psi(g_n) = z/g_n(z) = z - ((1-\alpha)/((n+1)(n+1-\alpha)))z^{n+1}$, $z \in U$.

Proof. Since $\psi(g) \in C(\alpha)$ it has the Taylor series expansion $\psi(g) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, $z \in U$. By the definition of $g(z)$,

$$b_n = \sum_{k=0}^{n-1} b_k \, a_{n-k+1} \tag{5.5.8}$$

for $n \geq 1$ where $b_o = 1$.

First we show that $\{b_n\}$ is a sequence of nonnegative real numbers.

It follows from the equation (5.5.8) that $b_1 = a_2 \geq 0$. Now assume that $b_k \geq 0$ for $1 \leq k \leq n$, $n \in \mathbb{N}$. Since, $b_{n+1} = \sum_{k=0}^{n} b_k \, a_{n+2-k}$ and a_k's are nonnegative, we have $b_{n+1} \geq 0$. This proves that $\{b_n\}$ is a sequence of nonnegative real numbers.

By the inequality (1.4.8), we have

$$b_1 = a_2 \leq \frac{1-\alpha}{2(2-\alpha)} .$$

This proves the inequality (5.5.7) for $n = 1$.

Now, let the inequaltiy (5.5.7) be true for n, satisfying $1 \leq n \leq k$, $k \in \mathbb{N}$. Then,

$$b_{k+1} = \sum_{n=0}^{k} b_n \, a_{k+2-n} \leq \sum_{n=0}^{k} \frac{1-\alpha}{(n+1)(n+1-\alpha)} a_{k+2-n}. \tag{5.5.9}$$

Set, for $n \geq 2$,

$$a_n = \lambda_n \frac{1-\alpha}{n(n-\alpha)} .$$

For $\psi(g) = z - \sum_{n=2}^{\infty} a_n z^n \in C(\alpha)$, it is necessary (cf. Section 1.4) that $\sum_{n=2}^{\infty} n(n-\alpha)\, a_n \le 1-\alpha$. Thus,

$$\lambda_n \ge 0$$

for $n \ge 2$ and

$$\sum_{n=1}^{k+1} \lambda_{n+1} \le 1. \tag{5.5.10}$$

The inequality (5.5.9) is equivalent to

$$\begin{aligned} b_{k+1} &\le \sum_{n=0}^{k} \lambda_{k+2-n} \frac{1-\alpha}{(n+1)(n+1-\alpha)} \cdot \frac{1-\alpha}{(k+2-n)(k+2-n-\alpha)} \\ &\le \frac{(1-\alpha)}{(k+2)(k+2-\alpha)} \sum_{n=0}^{k} \lambda_{k+2-n} \\ &\le \frac{(1-\alpha)}{(k+2)(k+2-\alpha)} . \end{aligned}$$

The second inequality holds since

$$(n+1)(n+1-\alpha)(k+2-n)(k+2-n-\alpha) \ge (1-\alpha)(k+2)(k+2-\alpha)$$

for $0 \le n \le k$ and the last inequality holds due to (5.5.10). This proves the inequality (5.5.7) for $n = k+1$ and the proof of the theorem is complete by the induction argument. It is easily seen

that sharpness of (5.5.7) is attained for the function $\psi(g_n)$ where g_n is as in the statement of the theorem.

CHAPTER VI

SUPPORT POINTS AND DISTORTION PROPERTIES OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

6.1 In this chapter we determine the support points, growth theorems and distortion theorems for the following classes of analytic functions:

<u>The Class</u> $T(b,d_2,B_k)$:

Let $b,\ d_2 > 0$ and $\{B_k\}_{k=2}^{\infty}$ be any sequence of positive numbers. Set,

$$(6.1.1)\qquad T(b,d_2,\{B_k\}) = \{f(z) = z-bz^2 - \sum_{k=3}^{\infty} a_k z^k \in T: \sum_{k=2}^{\infty} B_k a_{k+1} \le d_2 b\}.$$

For convenience in the presentation of our proofs, throughout in the sequel we use the notation $T(b,d_2,B_k) = T(b,d_2,\{B_k\})$.

Clearly, the class $T(b,d_2,B_k) \subseteq T$ and therefore, $0 \le b \le 1/2$ (cf. (1.4.2), [134]). Further, if $B_k \ge kd_{k+1}$, $k = 2,3,4,\ldots;$ where $\{d_k\}_{k=1}^{\infty}$ is a non-decreasing sequence of positive numbers, then the functions in the class $T(b,d_2,B_k)$ have univalent Gelfond-Leontev derivatives in the unit disc U (cf. (1.4.19)). Thus, in particular, if $B_k \ge k(k+1)$, $k = 2,3,4,\ \ldots,$ then the functions in the class $T(b,d_2,B_k)$ have univalent derivatives in U.

Throughout in the sequel, we denote,

(6.1.2) $\quad T(b,\{d_k\},B_k) = \{f \in T(b,d_2,\{B_k\}) : B_k \geq kd_{k+1},\ k \geq 2\}$,

where, $0 < b \leq 1/2$ and $\{d_k\}_{k=1}^{\infty}$ is a non-decreasing sequence of positive numbers. Thus, the inequality (1.4.19), gives that the class $T(b,\{d_k\},B_k) \subseteq T_1(D)$ and $T(b,\{k\},B_k) \subseteq T_1$ (cf. Definition 1.4.13).

It is easily seen that, for $0 < B_k \leq C_k$, $k \geq 2$, the class $T(b,d_2,C_k) \subseteq T(b,d_2,B_k)$. Similarly, if $d_2 \leq d_2^*$, then the class $T(b,d_2,B_k) \subseteq T(b,d_2^*,B_k)$.

For special choices of the parameters b,d_2 and B_k, the class $T(b,d_2,B_k)$ reduces to subclasses of certain known classes of functions. Thus, the class $T((1-\alpha)/(4-3\alpha),\ 2,\ (k+1-\alpha)/(1-\alpha))$ is the subclass of $T^*(\alpha)$ (cf. Section 1.4), consisting of functions f with fixed second coefficient $|f''(0)|/2! = (1-\alpha)/(4-3\alpha), 0 \leq \alpha < 1$. The class $T((1-\alpha)/2(3-2\alpha),\ 2,\ (k+1)(k+1-\alpha)/(1-\alpha))$ is the subclass of $C(\alpha)$ (cf. Section 1.4), consisting of functions f with fixed second coefficient $|f''(0)|/2! = (1-\alpha)/2(3-2\alpha), 0 \leq \alpha < 1$. In fact, for positive numbers b,d_2, with $d_2 b \leq 1$ and a sequence of positive numbers $\{B_k\}_{k=2}^{\infty}$, we have

(6.1.3) $\quad T(b,d_2,B_k) = F_p(\{B_{k-1}\}) \quad ; \quad p = 1-d_2 b$

where, $F_p(\{B_{k-1}\})$ is the class introduced by Ahuja and Silverman (cf. Section 1.4,[3]).

In the case $d_2 b \leq 1$, the extreme points, growth theorem and distortion theorem for the class $T(b,d_2,B_k)$ were found in [3].

But, when $d_2b > 1$, the extreme points, growth theorem and distortion theorem for the class $T(b,d_2,B_k)$ are not known. The techniques of Ahuja and Silverman [3] can be carried over to show that the results and their proofs of the extreme points, growth theorem and distortion theorem for the class $T(b,d_2,B_k)$ when $d_2b \le 1$, continue to hold good when $d_2b > 1$. However, determining the support points of the class $T(b,d_2,B_k)$ has not been attempted at all so far.

<u>The</u> <u>Class</u> $C(b,d_2,B_k)$:

Let $b,d_2 > 0$ and $\{B_k\}_{k=2}^{\infty}$ be a sequence of positive numbers. Set

(6.1.4) $\quad C(b,d_2,\{B_k\}) = \{ f \in C\colon zf' \in T(2b,d_2,2B_k/(k+1))\}.$

For convenience in the presentation of our proofs, throughout in the sequel we use the notation $C(b,d_2,B_k) = C(b,d_2,\{B_k\})$.

Clearly $C(b,d_2,B_k) \subseteq C$ and therefore, (cf.(1.4.8), [130]) $0 \le b \le 1/4$. Further, if $B_k \ge k^2 d_{k+1}$, $k = 2,3,4,\ldots;$ where $\{d_k\}_{k=1}^{\infty}$ is a non-decreasing sequence of positive numbers, then the functions in the class $C(b,d_2,B_k)$ have univalent convex Gelfond-Leontev derivatives in the unit disc U (cf. (1.4.20)). Thus, if $B_k \ge k^2(k+1)$, $k = 2,3,4, \ldots;$ then the functions in the class $C(b,d_2,B_k)$ have univalent convex derivatives in U.

Throughout in the sequel, we denote,

(6.1.5) $\quad C(b,\{d_k\},B_k) = \{f \in C(b,d_2,\{B_k\})\colon B_k \ge k^2 d_{k+1},\ k \ge 2\},$

where $b > 0$, $\{d_k\}_{k=1}^{\infty}$ is a non-decreasing sequence of positive

numbers and $\{B_k\}_{k=2}^{\infty}$ is a sequence of positive numbers. Thus, the inequality (1.4.20) gives that the class $C(b,\{d_k\},B_k) \subseteq C_1(D)$ and the class $C(b,\{k\},B_k) \subseteq C_1$ (cf. Definition 1.4.14).

In the case $d_2 b \leq 1$, the extreme points, growth theorem and distortion theorem for the class $C(b,d_2,B_k)$ are contained in the work of Ahuja and Silverman [3]. But, when $d_2 b > 1$, the extreme points, growth theorem and distortion theorem for the class $C(b,d_2,B_k)$ are not known. The techniques in [3] can be carried over to show that the statements and proofs of the extreme points, growth theorem and distortion theorem for the class $C(b,d_2,B_k)$ when $d_2 b \leq 1$, continue to hold good when $d_2 b > 1$. However, determining the support points of the class $C(b,d_2,B_k)$ has not been attempted at all so far.

<u>The</u> <u>Class</u> $A\{n,M_k\}$:

For a sequence $\{M_k\}_{k=n+1}^{\infty}$, $n = 1,2,\ldots,$ of nonzero complex numbers, set

$$(6.1.6) \qquad A\{n,\{M_k\}\} = \{f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in A : \sum_{k=n+1}^{\infty} M_k\, a_k \leq 1;$$

$$\text{for } a_k \neq 0,\ \arg a_k = -\arg M_k\}.$$

For convenience in the presentation of our proofs, throughout in the sequel we use the notation $A\{n,M_k\} = A\{n,\{M_k\}\}$.

It is easily seen that for M_k, N_k nonzero complex numbers, with $\arg M_k = \arg N_k$ and $|M_k| \geq |N_k|$ for $k \geq n+1$, the class $A\{n,M_k\} \subseteq A\{n,N_k\}$.

If $M_k \leq -k$, then by the inequality (1.2.6), the class

(6.1.7) $\qquad A\{n,M_k\} \subseteq T$

and, if $M_k \in \mathbb{C}$ and $|M_k| \geq k$ then, by the same inequality (1.2.6), the class

(6.1.8) $\qquad A\{n,M_k\} \subseteq S^*$.

For $M_k \in \mathbb{C}$ with $0 < |M_k| < k$; $k \geq n+1$, the class $A\{n,M_k\}$ may contain nonunivalent functions also; consider, for example, the function $g_n(z) = z+z^{n+1}/M_{n+1}$, $n \geq 1$.

From the definition of the class $A\{n,M_k\}$, M_k nonzero complex, it follows that if $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ is in $A\{n,M_k\}$, then

$$|a_k| \leq \frac{1}{|M_k|}, \qquad k \geq n+1,$$

and the functions $z + z^k/M_k$, $k \geq n+1$, show that this inequality is sharp.

With M_k, real and negative, the class $A\{n,M_k\}$ has been considered earlier by several workers. Thus, if $M_k < 0$, the class $A\{n,M_k\}$ reduces to the class $A(n,\{-M_k\})$ (cf. Section 1.4), introduced by Sekine [124]. For $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $0 \leq \mu \leq 1$, the class

$$A\left\{1, \frac{-k(1+\mu\beta)}{(1+\mu)\beta(1-\alpha)}\right\} = P^*(\alpha, \beta, \mu)$$

(cf. Section 1.7) was investigated by Owa and Aouf [92]. The classes

$$\bigcup_{\beta\in\mathbb{R}} A\{1, e^{i((k-1)\beta-\pi)}/(1-\alpha)\} = \mathcal{S}_\alpha ,$$

$$\bigcup_{\beta\in\mathbb{R}} A\{1, ke^{i((k-1)\beta-\pi)}/(1-\alpha)\} = R_\alpha$$

(cf. Section 1.4) for $0 \le \alpha < 1$, were studied by Srivastava and Owa [146]. The classes

$$\bigcup_{\beta\in\mathbb{R}} A\{1, ke^{i((k-1)\beta-\pi)}\} = SV,$$

$$\bigcup_{\beta\in\mathbb{R}} A\{1, (k-\alpha)e^{i((k-1)\beta-\pi)}/(1-\alpha)\} = SV^*(\alpha)$$

(cf. Section 1.4) were considered by Silverman [133] for $0 \le \alpha < 1$.

Further, for positive numbers b, d_2, B_k, where $k \ge 2$, $bd_2 \le 1$, $0 < b \le 1/2$, the class (cf. (6.1.1))

(6.1.9) $$T(b, d_2, B_k) = A\{1, -M_k\} \cap T(b)$$

where

(6.1.10) $$T(b) = \{f\in T\colon f(z) = z-bz^2- \ldots \text{in } U\},$$

$M_2 = (1-d_2b)/b$, and $M_k = B_{k-1}$, $k = 3,4,5, \ldots$

Similarly, for positive numbers b, d_2, B_k where $k \ge 2$, $bd_2 \le 1$, $0 < b \le 1/4$, the class (cf.(6.1.4))

$$C(b, d_2, B_k) = A\{1, -M_k\} \cap \mathcal{C} \cap T(b)$$

where, $\mathcal{C}$ is as in Section 1.4, $T(b)$ is as in (6.1.10),

$$M_2 = (1-d_2b)/b, \text{ and } M_k = B_{k-1}, \ k = 3,4,5, \ldots .$$

We denote

(6.1.11) $$A^*\{n, M_k\} = \{f \in A\{n, \{M_k\}\}\colon |M_k| \ge k\}.$$

Thus, from containment relation (6.1.8) we have that the class $A^*\{n, M_k\} \subseteq S^*$.

In Section 6.2, the extreme points of the class $A\{n,M_k\}$ are determined. In the same section, the class $A_o(n,B_k,z_o)$ consisting of functions with two fixed points 0 and z_o, $-1 < z_o < 1$, is introduced and its extreme points are determined. Section 6.3 contains the description of support points for the classes $T(b,\{d_k\},B_k)$, $C(b,\{d_k\},B_k)$ and $A^*\{n,M_k\}$. We also determine the support points of the class $A_o^*(n,B_k,z_o)$ consisting of functions having two fixed points 0, z_o in $-1 < z_o < 1$ and satisfying $B_k \geq k$ for $k \geq n+1$. The results in Section 6.4 are devoted to finding growth and distortion properties for the classes $A\{n,M_k\}$ and $A_o(n,B_k,z_o)$. In the final Section 6.5, bounds on b_2 are obtained when $z/(1+\sum_{n=1}^{\infty} b_n z^n)$ is in $T(b,d_2,B_k)$ or $C(b,d_2,B_k)$. Finally, in this section, the radii for starlikeness, convexity etc., in the class $A\{n,M_k\}$ are also determined.

6.2 It is easily seen that the class $A\{n,M_k\}$ is convex. The following result is needed to determine the extreme points of the class $A\{n,M_k\}$:

<u>Theorem</u> 6.2.1 <u>Let</u>

$$(6.2.1)\qquad \begin{cases} f_1(z) = z, \\ \\ f_k(z) = z + \dfrac{z^k}{M_k} \qquad (k \geq n+1). \end{cases}$$

<u>Then</u>, <u>a</u> <u>function</u> $f \in A\{n,M_k\}$, <u>if</u> <u>and</u> <u>only</u> <u>if</u>,

$$(6.2.2)\qquad f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z),$$

<u>where</u> $\lambda_1 \ge 0,\ \lambda_k \ge 0$ <u>for</u> $k \ge n+1,\ \lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1$

<u>and</u> $\limsup_{k\to\infty} (\lambda_k/|M_k|)^{1/k} \le 1.$

<u>Proof</u>. Let a function $f(z)$ be expressed as in (6.2.2). Then,

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z)$$

$$= z + \sum_{k=n+1}^{\infty} \frac{\lambda_k}{M_k} z^k$$

$$\equiv z + \sum_{k=n+1}^{\infty} A_k z^k .$$

The limit superior condition gives that $f(z)$ is analytic in U. Further,

$$\arg A_k = -\arg M_k$$

for $A_k \neq 0$, $k \ge n+1$, and

$$\sum_{k=n+1}^{\infty} M_k A_k = \sum_{k=n+1}^{\infty} \lambda_k = 1-\lambda_1 \le 1.$$

Hence, $f \in A\{n,M_k\}$.

Conversely, let a function

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \tag{6.2.3}$$

be in $A\{n,M_k\}$. Thus,

$$\sum_{k=n+1}^{\infty} M_k a_k \le 1. \tag{6.2.4}$$

Put $\lambda_k = M_k a_k$ for $k \ge n+1$ and

$$\lambda_1 = 1 - \sum_{k=n+1}^{\infty} \lambda_k$$

Thus, from the inequality (6.2.4), we have

$$\lambda_1 \geq 0,\ \lambda_k \geq 0 \text{ for } k \geq n+1;\ \lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1.$$

Further, using (6.2.3),

$$f(z) = \lambda_1\ f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k\ (z + \frac{z^k}{M_k})$$

$$= \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k\ f_k(z).$$

This completes the proof of the converse part, and the proof of the theorem is complete.

<u>Remark</u> In [93], Owa et al. proved that for $M_k < 0$, $f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z)$ where $\lambda_1 \geq 0$; $\lambda_k \geq 0$ for $k \geq n+1$ and $\lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1$, if and only if, $f \in A\{n,M_k\}$. However, a condition that $f(z)$ is analytic in U is additionally required for the validity of 'if' part of the result of Owa et al. as shown in the proof of Theorem 6.2.1.

<u>Theorem</u> 6.2.2 <u>The extreme points of the class</u> $A\{n,M_k\}$ <u>are the functions</u> f_1 <u>and</u> f_k, $(k \geq n+1)$ <u>in the class</u> $A\{n,M_k\}$.

<u>Proof</u>. Clearly, the functions f_1, f_k, $(k \geq n+1)$ are in the class $A\{n,M_k\}$. First we show that the functions f_1 and f_k $(k \geq n+1)$ are extreme points of the class $A\{n,M_k\}$.

Suppose that the function f_1 is not an extreme point of $A\{n,M_k\}$. Thus there exist distinct functions $g_i(z) = z + \sum_{k=n+1}^{\infty} a_{k,i} z^k$ in $A\{n,M_k\}$, $i = 1,2$, and $0 < t < 1$ such that

$$f_1(z) = tg_1(z) + (1-t)\, g_2(z), \qquad z \in U. \tag{6.2.5}$$

By comparing the coefficients of z^k $(k \geq n+1)$ on both sides of the equation (6.2.5), we have

$$ta_{k,1} + (1-t)a_{k,2} = 0,$$

or

$$a_{k,1} = a_{k,2} = 0.$$

Hence, $g_1(z)$ and $g_2(z)$ are identical which is a contradiction. Thus, f_1 is an extreme point of the class $A\{n,M_k\}$.

Similarly, it can be easily verified that the functions f_k $(k \geq n+1)$ are extreme points of the class $A\{n,M_k\}$.

Conversely, let a function g, distinct from f_1, f_k $(k \geq n+1)$, be an extreme point of the class $A\{n,M_k\}$. Theorem 6.2.1 gives that there exist $\lambda_1, \lambda_k \geq 0$, $k \geq n+1$, such that

$\lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1$ and for $z \in U$

$$g(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k\, f_k(z). \tag{6.2.6}$$

Since the function g is distinct from f_1, f_k $(k \geq n+1)$, there exists an m, $m = 1$ or $m \geq n+1$ such that $0 < \lambda_m < 1$. First let $m = 1$. Then (cf. (6.2.6)), for $z \in U$,

$$\begin{aligned} g(z) &= \lambda_1 f_1(z) + (1-\lambda_1) \sum_{k=n+1}^{\infty} \frac{\lambda_k}{1-\lambda_1} f_k(z) \\ &\equiv \lambda_1 f_1(z) + (1-\lambda_1)\, h(z) \end{aligned}$$

where, $h \in A\{n,M_k\}$ (cf. Theorem 6.2.1) and f_1, h are distinct. This contradicts that the function g is an extreme point of the

class $A\{n,M_k\}$. A contradiction is similarly arrived at in the case $m \geq n+1$. Thus the proof of the theorem is complete.

Remarks 1. Choosing $n = 1$ and $M_k = -k(1+\mu\beta)/(1+\mu)\beta(1-\alpha)$, Theorems 6.2.1 and 6.2.2 give the analogous results for the class $A\{n,M_k\} = P^*(\alpha, \beta, \mu)$ (cf. Section 1.7) found by Owa and Aouf [92] where, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 \leq \mu \leq 1$ and $k \geq 2$.

2. Choosing $M_k < 0$, Theorems 6.2.1 and 6.2.2 give the corresponding results for the class $A\{n,-M_k\} = A(n, \{-M_k\})$ (cf. Section 1.4) found by Owa et al. [93] where $n = 1,2,3, \ldots$

3. For a sequence $B = \{\tilde{B}_k\}_{k=n+1}^{\infty}$ consisting of subsets of nonzero complex numbers such that each $\tilde{B}_k$, $k \geq n+1$, is contained in a ray $z = te^{i\theta_k}$, $0 < t < \infty$, define

(6.2.7) $$\mathcal{A}(n,B) = \{f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{A} : \exists \text{ a sequence } \{B_k\}_{k=n+1}^{\infty},$$
$$B_k \in \tilde{B}_k;\ \text{for } a_k \neq 0,\ \arg a_k = -\arg B_k;\ \sum_{k=n+1}^{\infty} B_k a_k \leq 1\},$$

where, $n = 1,2,3, \ldots$

The class $\mathcal{A}(n,B)$ is more general than the class $A\{n,M_k\}$ (cf. (6.1.6)). In fact, for $B = \{\tilde{B}_k\}_{k=n+1}^{\infty} \equiv \{\{M_k\}\}_{k=n+1}^{\infty}$, the class $\mathcal{A}(n,B) = A\{n,M_k\}$. The class $\mathcal{A}(n,B)$ is convex. It is not known whether results analogous to Theorems 6.2.1 and 6.2.2 hold for the class $\mathcal{A}(n,B)$.

Now, we define a class of functions analytic in U, that have two fixed points in U and determine the extreme points and support points of this class.

<u>The Class</u> $A_0(n, B_k, z_0)$:

Let, for $-1 < z_0 < 1$,

$$\mathcal{A}(z_0) = \{f \in \mathcal{A}:\ f(z_0) = z_0,\ f'(0) \neq 0\}.$$

For $B_k > |z_0|$, $k \geq n+1$, $n = 1,2,3, \ldots,$ define

$$A_0(n,\ B_k,\ z_0) = \{f(z) = a_1 z - \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{A}(z_0): \tag{6.2.8}$$

$$a_1 > 0,\ a_k \geq 0,\ k \geq n+1,\ \sum_{k=n+1}^{\infty} B_k a_k \leq a_1\}.$$

The class $A_0(n, B_k,\ 0) \cap \{f \in \mathcal{A}:\ f'(0) = 1\}$ is same as the class $A(n, \{-B_k\})$ (cf. Section 1.4), $B_k > 0$, of Sekine studied in [124]. The classes

$$A_0\ (1,\ \frac{k-\alpha}{1-\alpha},\ z_0) = S_0^*(\alpha,\ z_0)$$

and

$$A_0\ (1,\ \frac{k(k-\alpha)}{1-\alpha}\ ,\ z_0) = K_0(\alpha,\ z_0)$$

were studied by Silverman (cf. Section 1.4, [131]), where $0 \leq \alpha < 1$ and $-1 < z_0 < 1$, $z_0 \neq 0$. The classes

$$A_0\ (n,\ (k(B+1)-(A+1))/(B-A), z_0) = S_1(A, B, z_0),$$

$$A_0\ (n, k(k(B+1)-(A+1))/(B-A), z_0) = K_1(A, B, z_0).$$

(cf. Section 1.4) were considered by Lakshma Reddy and Padmanabhan [68] where, $-1 \leq A < B \leq 1$. For $k = 2,3,\ldots,$ $0 \leq c_1/d_1 \leq \alpha$,

$$A_0\ (1,\ (c_k-\alpha d_k)/(c_1-\alpha d_1), z_0) = F[s, g, \alpha, z_0]$$

(cf. Section 1.4) was investigated by Mishra and Sahu [85] where

$s(z) = \sum_{k=1}^{\infty} c_k z^k$ and $g(z) = \sum_{k=1}^{\infty} d_k z^k$ are in A, $(c_k/c_1)-(d_k/d_1) > 0$.

Further, let $-1 < z_o < 1$, $n = 1,2,3, \ldots$, $B_k > 0$, $k \geq n+1$, and $f \in A(z_o)$. Then,

(6.2.9) $\quad f \in A_o(n,B_k,z_o)$, if and only if, $f/f'(0) \in A\{n, -B_k\}$.

It follows from the definition of the class, $A_o(n,B_k,z_o)$ that if $f(z) = a_1 z - \sum_{k=n+1}^{\infty} a_k z^k \in A_o(n,B_k,z_o)$, then

$$0 \leq a_k \leq a_1/B_k, \quad k \geq n+1$$

and the functions $(B_k z-z^k)/(B_k-z_o^{k-1})$, $k \geq n+1$, show that the upper bound of a_k's is sharp, for each $k \geq n+1$.

It is easily seen that the class $A_o(n,B_k,z_o)$ is convex. The following result is needed to determine the extreme points of the class $A_o(n,B_k,z_o)$:

<u>Theorem</u> 6.2.3 <u>Let</u> $B_k > |z_o| > 0$, $k \geq n+1$, $n = 1,2,3, \ldots$, $-1 < z_o < 1$,

$$\text{(6.2.10)} \quad \begin{cases} f_1(z) = z, \\ f_k(z) = \dfrac{B_k}{B_k-z_o^{k-1}} z - \dfrac{1}{B_k-z_o^{k-1}} z^k, & k \geq n+1. \end{cases}$$

<u>Then, a function</u> $f \in A_o(n,B_k,z_o)$, <u>if and only if</u>,

$$\text{(6.2.11)} \quad f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z),$$

<u>where</u> $\lambda_1 \geq 0$, $\lambda_k \geq 0$, $k \geq n+1$, $\lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1$ <u>and</u>

$\limsup_{k \to \infty} (\lambda_k/(B_k - z_o^{k-1}))^{1/k} \leq 1$.

<u>Proof.</u> Let $f(z) = a_1 z - \sum_{k=n+1}^{\infty} a_k z^k \in A_o(n, B_k, z_o)$. Then, $\sum_{k=n+1}^{\infty} B_k a_k \leq a_1$. Thus, $\lambda_k \equiv a_k (B_k - z_o^{k-1}) \geq 0$ for $k \geq n+1$. Set $\lambda_1 = 1 - \sum_{k=n+1}^{\infty} \lambda_k$. Then, $\lambda_1 \geq 0$ and $f(z)$ is of the form (6.2.11). The proof of the converse part is similar to that of the 'if' part of Theorem 6.2.1 and is omitted.

<u>Theorem</u> 6.2.4 <u>The extreme points of the class</u> $A_o(n, B_k, z_o)$ <u>with</u> n, B_k, z_o <u>as in</u> Theorem 6.2.3, <u>are the functions</u> f_1 <u>and</u> f_k $(k \geq n+1)$, <u>defined in</u> (6.2.10).

<u>Proof</u>. The proof of the theorem is similar to that of Theorem 6.2.2 and is omitted.

<u>Remarks</u> 1. Choosing $n = 1$, and $B_k = (k-\alpha)/(1-\alpha)$, $0 \leq \alpha < 1$, Theorems 6.2.3 and 6.2.4 give the anologous results for the class $A_o(1, (k-\alpha)/(1-\alpha), z_o) = S_o^*(\alpha, z_o)$ (cf. Section 1.4) found earlier by Silverman [131], where $-1 < z_o < 1$ and $z_o \neq 0$.

2. Choosing $n = 1$ and $B_k = k(k-\alpha)/(1-\alpha)$, $0 \leq \alpha < 1$, Theorems 6.2.3 and 6.2.4 give the analogous results for the class $A_o(1, k(k-\alpha)/(1-\alpha), z_o) = K_o(\alpha, z_o)$ (cf. Section 1.4) found earlier by Silverman [131], where $-1 < z_o < 1$, $z_o \neq 0$.

3. Choosing $B_k = (k(B+1) - (A+1))/(B-A)$, $-1 \leq A < B \leq 1$, Theorem 6.2.3 gives the analogous result for the class

$A_o(n,B_k,z_o) = S_1(A,B,z_o)$ (cf. Section 1.4) found earlier by Lakshma Reddy and Padmanabhan [68]. Theorem 6.2.4 gives the extreme points of $S_1(A,B,z_o)$.

4. Choosing $B_k = k(k(B+1)-(A+1))/(B-A)$, $-1 \le A < B \le 1$, Theorem 6.2.3 gives the corresponding result for the class $A_o(n,B_k,z_o) = K_1(A,B,z_o)$ (cf. Section 1.4) found earlier by Lakshma Reddy and Padmanabhan [68]. Theorem 6.2.4 gives the exteme points of $K_1(A,B,z_o)$.

5. Choosing $B_k = (c_k-\alpha\, d_k)/(c_1-\alpha\, d_1)$, where $s(z) = \sum_{k=1}^{\infty} c_k z^k$ and $g(z) = \sum_{k=1}^{\infty} d_k z^k$ are in A, $(c_k/c_1)-(d_k/d_1) > 0$, $0 \le \alpha \le (c_1/d_1)$. Theorems 6.2.3 and 6.2.4 give the analogous results for the class $F\,[s,g,\alpha,z_o]$ (cf. Section 1.4), z_o real, $0 < |z_o| < 1$ obtained earlier by Mishra and Sahu [85].

6.3 We begin with determining the set $Supp\{T(b,\{d_k\},B_k)\}$ consisting of the support points of the class $T(b,\{d_k\},B_k)$.

<u>Theorem</u> 6.3.1 <u>Let</u> $0 < b < 1/2$ <u>and</u> $B_k \ge (k+1)d_2 b/(1-2b)$, $k \ge 2$. <u>Then</u>,

(6.3.1) $$Supp\{T(b,\{d_k\},B_k)\} = \{f \in T\,(b,\{d_k\},B_k): f(z) = z-bz^2 - d_2 b \sum_{k=2}^{\infty} \frac{\lambda_k}{B_k} z^{k+1},\ \lambda_k \ge 0,\ \sum_{k=2}^{\infty} \lambda_k \le 1 \text{ and } \lambda_j = 0 \text{ for some } j\ \}.$$

<u>Proof</u>. Let the function f_o be in the class $T(b,\{d_k\},B_k)$ and let

$$f_o(z) = z - bz^2 - d_2 b \sum_{k=2}^{\infty} \frac{\lambda_k}{B_k} z^{k+1}, \qquad z \in U$$

where $\lambda_k \geq 0$, $\sum_{k=2}^{\infty} \lambda_k \leq 1$, and $\lambda_j = 0$ for some $j \geq 2$.

If $b_1 = b_2 = b_{j+1} = 1$ and $b_k = 0$ for $k \geq 3$, $k \neq j+1$, then

$$\limsup_{k \to \infty} |b_k|^{1/k} < 1.$$

Then, with the help of (1.7.2), we define the continuous linear functional $J(f)$ given by the sequence $\{b_k\}_{k=1}^{\infty}$. Thus, $J(f_o) = 1-b$ and that $J(f) = 1-b-\mu_j d_2 b/B_j$ for all

$f(z) = z-bz^2 - d_2 b \sum_{k=2}^{\infty} \mu_k z^{k+1}/B_k \in T(b,\{d_k\},B_k)$, with $\mu_k \geq 0$ and $\sum_{k=2}^{\infty} \mu_k \leq 1$. Now, it follows that $\mathrm{Re}\{J(f)\} \leq \mathrm{Re}\{J(f_o)\}$ for all $f \in T(b,\{d_k\},B_k)$ since $f \in T(b,\{d_k\},B_k)$, if and only if,

$f(z) = z - bz^2 - d_2 b \sum_{k=2}^{\infty} \mu_k z^{k+1}/B_k$, where $\mu_k \geq 0$, $\sum_{k=2}^{\infty} \mu_k \leq 1$. Further, since $f_j(z) = z-bz^2 - d_2 b z^{j+1}/B_j \in T(b,\{d_k\},B_k)$, we have that $J(f_j) = 1-b-d_2 b/B_j < J(f_o)$. This implies that f_o belongs to $\mathrm{Supp}\{T(b,\{d_k\},B_k)\}$.

Conversely, let $f_o \in \mathrm{Supp}\{T(b,\{d_k\},B_k)\}$. Then, there exists a continuous linear functional J on $\mathcal{A}$ such that $\mathrm{Re}\{J(f)\}$ is non-constant on the class $T(b,\{d_k\},B_k)$ and

$$\mathrm{Re}\,\{J(f_o)\} = \max\,\{\mathrm{Re}\{J(f)\}: \ f \in T(b,\{d_k\},B_k)\}.$$

Define the class

$$G \equiv \{f \in T(b,\{d_k\},B_k) : \mathrm{Re}\,\{J(f)\} = \max\{\mathrm{Re}\{J(g)\}: g \in T(b,\{d_k\},B_k)\} = \mathrm{Re}\,J\,(f_o)\}.$$

The class G is closed, convex and locally uniformly bounded. Thus G is compact. Since $f_o \in G$, by Krein-Mil'man Theorem, Ext $\{G\}$ is

nonempty. The functions f and $g \in T(b,\{d_k\},B_k)$ and, for $0 < t < 1$, the function $tf + (1-t)g \in G$, we have that f and g are in G. So $\mathrm{Ext}\{G\} \subseteq \mathrm{Ext}\{T(b,\{d_k\},B_k)\} = \{f_1(z) = z-bz^2\} \cup \{f_k:$

$$f_k(z) = z - bz^2 - d_2b\, z^{k+1}/B_k,\ k \geq 2\}.$$

Since $\mathrm{Re}\{J(f)\}$ is non-constant on the class $T(b,\{d_k\},B_k)$, we have that $\mathrm{Ext}\{G\} \subsetneq \mathrm{Ext}\{T(b,\{d_k\},B_k)\}$. Thus there exists an integer $j \geq 1$ such that the extreme point of $T(b,\{d_k\},B_k)$ f_j does not belong to $\mathrm{Ext}\{G\}$. Let n be the smallest of such j's.

Suppose $n > 1$. In this case, $f_o \in G$ gives that

$$f_o(z) = z-bz^2 - d_2b \sum_{\substack{k=2\\k\neq n}}^{\infty} \frac{\lambda_k}{B_k} z^{k+1}$$

where $\lambda_k \geq 0$ and $\sum_{\substack{k=2\\k\neq n}}^{\infty} \lambda_k \leq 1$. Thus f_o is in the class on the right hand side of the equality (6.3.1).

Now, consider $n = 1$. Here,

$$\mathrm{Ext}\{G\} \subseteq \{f_j \in \mathrm{Ext}\{T(b,\{d_k\},B_k)\}:\ j \geq 2\}.$$

Since B_k tends to ∞ as k goes to ∞, and G is closed, there are only a finite number of f_k's in $\mathrm{Ext}\{G\}$. Hence, for $z \in U$, $f_o(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$, $\exists\ \lambda_k \geq 0$, $\sum_{k=1}^{\infty} \lambda_k = 1$, and the fact that f_o is in G, gives that f_o is in the class on the right hand side of the equality (6.3.1). Thus the proof of the theorem is complete.

<u>Remark</u>. For b, $d_2 > 0$, $d_2b \leq 1$ and $B_k \geq (k+1)d_2b/(1-2b) > 0$, $k = 2,3,\ldots$; Theorem 6.3.1 gives the support points of the class

$T(b,\{d_k\},B_k) = F_p(\{B_{k-1}\})$, $p = 1-d_2b$ where $F_p(\{B_{k-1}\})$ is the class introduced by Ahuja and Silverman (cf. Section 1.4, [3]).

The following result gives the set, $\mathrm{Supp}\{C(b,\{d_k\},B_k)\}$ consiting of the support points of the class $C(b,\{d_k\},B_k)$.

Theorem 6.3.2 Let $0 < b < 1/4$ and $B_k \geq (k+1)^2 d_2b/(1-4b)$ for $k \geq 2$. Then ,

$$\mathrm{Supp}\{C(b,\{d_k\},B_k\} = \{f \in C(b,\{d_k\},B_k):$$

$$f(z) = z-bz^2-d_2b \sum_{k=2}^{\infty} \frac{\lambda_k}{B_k} z^{k+1},$$

$$\lambda_k \geq 0, \sum_{k=2}^{\infty} \lambda_k \leq 1, \lambda_j = 0 \text{ for some } j\}.$$

Proof. The proof of the theorem is similar to that of Theorem 6.3.1 and is omitted.

While finding the support points of the class $A^*\{n,M_k\}$ with $M_k < 0$, Owa et al.[93] used arguments wherein it is essential to show that $\mathrm{Ext}\{G_j\} \subseteq \{f_k : f_k \in \mathrm{Ext}\{A^*\{n,M_k\}\}, k \geq n+1, k \neq j\}$ where, $G_j \equiv \{f_k \in \mathrm{Ext}\ \{A^*\{n,M_k\}\}: \mathrm{Re}\ J(f_o) = \mathrm{Re} J(f_k)\}$ for f_o in $\mathrm{Supp}\{A^*\ \{n,M_k\}\}$ and some $j \geq n+1$. Though the result of Owa et al. is true, the arguments in their proof contain an error since the above containment relation is not true in general as shown by the following example:

The continuous linear functional

$$J(f) \equiv a_1 + \frac{1}{2} a_{n+1} \text{ for } f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{A}$$

gives that $f_1(z) \equiv z \in \mathrm{Supp}\{A^*\{n,M_k\}\}$ for $M_k < 0$, $f_1 \in \mathrm{Ext}\{G_j\}$

but $f_1 \notin \{f_k : f_k \in \mathrm{Ext}\{A^*\{n,M_k\}\} : k \geq n+1,\ k \neq j\}$ for any $j \geq n+1$ such that $\mathrm{Re}\ J(f_j) < \mathrm{Re}\ J(f_1)$. There does exist at least one $j \geq n+1$, for instance, $j = n+1$, such that $\mathrm{Re}\ J(f_1) > \mathrm{Re}\ J(f_j)$.

In the following result the set $\mathrm{Supp}\{A^*\{n,M_k\}\}$ consisting of the support points of the class $A^*\{n,M_k\}$ for any arbitrary sequence $\{M_k\}_{k=n+1}^{\infty}$ of nonzero complex numbers is determined. The theorem of Owa et al. (Theorem 3, [93]) follows as a particular case from our result and the arguments in the proof of our theorem correct the erraneous arguments in that of Owa et al.

<u>Theorem</u> 6.3.3

$$(6.3.2) \qquad \mathrm{Supp}\{A^*\{n,M_k\}\} = \{f \in A^*\{n,M_k\} : f(z) = z + \sum_{k=n+1}^{\infty} \frac{\lambda_k}{M_k} z^k,$$

$$\lambda_k \geq 0,\ \sum_{k=n+1}^{\infty} \lambda_k \leq 1,\ \lambda_j = 0 \text{ \underline{for} \underline{some} } j\ \}.$$

<u>Proof</u>. Let a function $f(z) = z + \sum_{k=n+1}^{\infty} \frac{\lambda_k}{M_k} z^k \in A^*\{n,M_k\}$ with $\lambda_k \geq 0$, $\sum_{k=n+1}^{\infty} \lambda_k \leq 1$ and $\lambda_j = 0$ for some j. Define

$$b_1 = 1,\ b_j = -M_j/|M_j|\ ,\ b_k = 0 \text{ for } k \geq n+1,\ k \neq j.$$

Now by setting $J(g) = \sum_{k=0}^{\infty} a_k b_k$ for $g(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{A}$, (1.7.2) gives that J is a continuous linear functional on $\mathcal{A}$. Now $J(f) = 1$ and $J(f_j) = 1 - 1/|M_j| < J(f)$ where $f_j(z) = z + \frac{z^j}{M_j}$ is in $A^*\{n,M_k\}$. Thus $\mathrm{Re}\{J\}$ is non-constant on $A^*\{n,M_k\}$. For the function $g(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ belonging to $A^*\{n,M_k\}$, we have $J(g) = 1 - |a_j| \leq J(f)$. Thus $f \in \mathrm{Supp}\{A^*\{n,M_k\}\}$.

Conversely, let $f \in \mathrm{Supp}\{A^*\{n,M_k\}\}$. Then there exists a continuous linear functional J on $\mathcal{A}$ such that $\mathrm{Re}\{J\}$ is non-constant on $A^*\{n,M_k\}$ and $\mathrm{Re}\{J(f)\} \geq \mathrm{Re}\{J(g)\}$ for all functions $g \in A^*\{n,M_k\}$. Define

$$G = \{h \in A^*\{n,M_k\}: \mathrm{Re}\{J(h)\} = \max_{\mathcal{f}\in A^*\{n,M_k\}} \mathrm{Re}\ (J(\mathcal{f})) = \mathrm{Re}\ \{J(f)\}\}.$$

The function $f \in G$ and so G is non-empty. In addition, G is closed, convex and locally uniformly bounded. Thus, G is compact. Now by the Krein-Mil'man theorem $\mathrm{Ext}\{G\}$ is non-empty. When $0 < t < 1$ and $tg + (1-t)h \in G$ with functions g and h in $A^*\{n,M_k\}$, we have the functions g and h are in G also. So $\mathrm{Ext}\{G\}$ is contained in $\mathrm{Ext}\{A^*\{n,M_k\}\}$.

Since $\mathrm{Re}\{J\}$ is non-constant on $A^*\{n,M_k\}$, we have $\mathrm{Ext}\{G\}$ is strictly contained in $\mathrm{Ext}\{A^*\{n,M_k\}\}$. Thus, there exists an integer $j \geq 1$ such that $f_j \notin \mathrm{Ext}\{G\}$. Let m be the smallest of such j's.

When $m > 1$, we have

$$\mathrm{Ext}\ \{G\} \subseteq \{f_i \in \mathrm{Ext}\ \{A^*\{n,M_k\}\}: i \geq n+1 \text{ and } i \neq m\} \cup \{f_1\}.$$

Thus from Theorem 6.2.1, it follows that, the function

$$f(z) = z + \sum_{k=n+1}^{\infty} \frac{\lambda_k}{M_k} z^k \text{ with } \lambda_k \geq 0,\ \sum_{k=n+1}^{\infty} \lambda_k \leq 1, \lambda_m = 0. \text{ Hence, } f$$

is in the class on the right hand side of the equality (6.3.2).

When $m = 1$, we have

$$\mathrm{Ext}\{G\} \subseteq \{f_i \in \mathrm{Ext}\{A^*\{n,M_k\}\}: i \geq n+1\}.$$

Theorem 6.3.3.

6. Choosing $M_k = (k-\alpha)e^{i((k-1)\beta-\pi)}/(1-\alpha)$, $0 \le \alpha < 1, \beta \in \mathbb{R}$, Theorems 6.2.1 and 6.2.2 give the analogous results for the class $A(n,M_k) = SV^*(\theta_n,\beta)$, (cf. Section 1.4) obtained earlier by Silverman [133] and the support points of $SV^*(\theta_n,\beta)$ are described by Theorem 6.3.3.

For $n = 1,2,3, \ldots, -1 < z_0 < 1$, $z_0 \ne 0$, denote

$$A_0^*(n,B_k,z_0) = \{f \in A(n,B_k,z_0) : B_k \ge k,\ k \ge n+1\}.$$

The follwing theorem gives the set, Supp $\{A_0^*(n,B_k,z_0)\}$ consisting of the support points of the class $A_0^*(n,B_k,z_0)$.

<u>Theorem</u> 6.3.4 <u>Let</u> $n = 1,2,3,\ldots;\ -1 < z_0 < 1$, $z_0 \ne 0$. <u>Then</u>,

$$\mathrm{Supp}\{A_0^*(n,B_k,z_0)\} = \{f \in A_0^*(n,B_k,z_0) : f(z) = f'(0)z - f'(0)\sum_{k=n+1}^{\infty} \frac{\lambda_k}{B_k} z^k,$$

$$\lambda_k \ge 0,\ \sum_{k=n+1}^{\infty} \lambda_k \le 1 \text{ and } \lambda_j = 0 \text{ for some } j\}.$$

<u>Proof</u>. The proof of the theorem is analogous to that of Theorem 6.3.3 and is omitted.

<u>Remarks</u>. 1. Choosing $n = 1$, and $B_k = (k-\alpha)/(1-\alpha)$, $0 \le \alpha < 1$, Theorem 6.3.4 gives the support points of the class $A_0(1,(k-\alpha)/(1-\alpha),z_0) = S_0^*(\alpha,z_0)$ (cf. Section 1.4) where $-1 < z_0 < 1$, $z_0 \ne 0$.

2. Choosing $n = 1$, and $B_k = k(k-\alpha)/(1-\alpha)$, $0 \le \alpha < 1$, Theorem 6.3.4 gives the support points of the class $A_0(1,k(k-\alpha)/(1-\alpha),z_0) = K_0(\alpha,z_0)$ (cf. Section 1.4) where $-1 < z_0 < 1$,

$z_o \neq 0$.

3. Choosing $B_k = (k(B+1)-(A+1))/(B-A)$, $-1 \le A < B \le 1$, Theorem 6.3.4 gives the support points of the class $A_o(n,B_k,z_o) = S_1(A,B,z_o)$ (cf. Section 1.4) which was introduced by Lakshma Reddy and Padmanabhan [68].

4. Choosing $B_k = k(k(B+1)-(A+1))/(B-A)$, $-1 \le A < B \le 1$, Theorem 6.3.4 gives the support points of the class $A_o(n,B_k,z_o) = K_1(A,B,z_o)$ (cf. Section 1.4) which was introduced by Lakshma Reddy and Padmanabhan [68].

6.4 In this section growth and distortion properties of functions in the classes $A\{n,M_k\}$ and $A_o(n,B_k,z_o)$ are studied.

<u>Theorem</u> 6.4.1 <u>If</u> $f \in A\{n,M_k\}$ <u>and</u> $|M_k| \le |M_{k+1}|$, $k \ge n+1$, <u>then</u>

$$\text{Max}\left(0, r - \frac{r^{n+1}}{|M_{n+1}|}\right) \le |f(z)| \le r + \frac{r^{n+1}}{|M_{n+1}|}, \quad |z| = r \tag{6.4.1}$$

<u>in the unit disc</u> U. <u>The inequality is sharp for</u> $|M_{n+1}| \ge 1$.

<u>Proof</u>. By Theorem 6.2.1, a function $f \in A\{n,M_k\}$ can be written as

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+1}^{\infty} \lambda_k f_k(z), \qquad z \in U \tag{6.4.2}$$

where $\lambda_1 \ge 0$, $\lambda_k \ge 0$, $k \ge n+1$, $\lambda_1 + \sum_{k=n+1}^{\infty} \lambda_k = 1$ and the functions $f_1(z)$, $f_k(z)$ $(k \ge n+1)$ are as in (6.2.1). Thus

$$r\left(1 - \frac{r^n}{|M_{n+1}|} \sum_{k=n+1}^{\infty} \lambda_k\right) \le |f(z)| \le r\left(1 + \frac{r^n}{|M_{n+1}|} \sum_{k=n+1}^{\infty} \lambda_k\right)$$

$$r - \frac{r^{n+1}}{|M_{n+1}|} \le |f(z)| \le r + \frac{r^{n+1}}{|M_{n+1}|}$$

which gives the inequality (6.4.1). The function $z+z^{n+1}/M_{n+1}$ gives equality on the right hand side of (6.4.1) at the point $z = |z|\, e^{i(\arg M_{n+1})/n}$. The same function gives equality on the left hand side of (6.4.1) at $z = |z| e^{i(\arg M_{n+1}+\pi)/n}$ when $|M_{n+1}| \geq 1$.

<u>Remarks</u>. 1. It follows from Theorem 6.4.1 that for the class $A\{n,M_k\}$ with $n+1 \leq |M_k| \leq |M_{k+1}|$, $k \geq n+1$, $n = 1,2,3, \ldots$, the Koebe domain is $\{ w \in \mathbb{C} : |w| < (1-1/|M_{n+1}|)\}$.

2. The Growth theorem for the class

$$\bigcup_{\beta \in \mathbb{R}} A\{1, e^{i((k-1)\beta-\pi)}/(1-\alpha)\} = \mathcal{B}_\alpha,$$

(cf. Section 1.4) $0 \leq \alpha < 1$, found earlier by Srivastava and Owa [146], follows from Theorem 6.4.1.

3. The Growth theorem for the class

$$\bigcup_{\beta \in \mathbb{R}} A\{1, ke^{i((k-1)\beta-\pi)}/(1-\alpha)\} = R_\alpha$$

(cf. Section 1.4) found earlier by Srivastava and Owa [146] follows from Theorem 6.4.1, when $0 \leq \alpha < 1$.

4. The Growth theorems for the classes

$$\bigcup_{\beta \in \mathbb{R}} A\{1, ke^{i((k-1)\beta-\pi)}\} = SV$$

and

$$\bigcup_{\beta \in \mathbb{R}} A\{1, (k-\alpha)e^{i((k-1)\beta-\pi)}/(1-\alpha)\} = SV^*(\alpha)$$

(cf. Section 1.4) $0 \leq \alpha < 1$ found earlier by Silverman [133] follow from Theorem 6.4.1.

Next we determine the bounds on the growth of functions in the class $A_o(n,B_k,z_o)$ (cf. (6.2.8)).

<u>Theorem</u> 6.4.2 <u>If</u> $f \in A_o(n,B_k,z_o)$, $B_{k+1} \geq B_k > |z_o| > 0$, $k \geq n+1$, $n = 1,2,3,\ldots$, $-1 < z_o < 1$, <u>then</u>,

$$(6.4.3) \qquad \text{Max}\ (0,\ f'(0)\ (r - \frac{r^{n+1}}{B_{n+1}})) \leq |f(z)| \leq f'(0)\ (r + \frac{r^{n+1}}{B_{n+1}})$$

<u>in the disc</u> U <u>where</u> $|z| = r$. <u>The inequality is sharp for</u> $B_{n+1} \geq 1$.

<u>Proof</u>. The theorem follows from the relation (6.2.9) and Theorem 6.4.1. The function f_{n+1} (cf. (6.2.10)) gives sharpness on the right hand side of (6.4.3) at $z = |z|e^{i\pi/n}$. The same function gives sharpness on the left hand side of (6.4.3) at $z=|z|$ when $B_{n+1} \geq 1$.

<u>Remark</u>. For $f \in S_1(A,B,z_o)$ or $K_1(A,B,z_o)$ (cf. Section 1.4), the upper bound in (6.4.3) is better than that found by Lakshma Reddy and Padmanabhan [68]. The lower bound in (6.4.3) is smaller than that found in [68]. For the sharp function the upper bound in (6.4.3) and that found in [68] are same. Similarly, for the sharp function the lower bound in (6.4.3) and that obtained in [68] are same.

Next, the bounds on distortion of functions in the class $A\{n,M_k\}$ are found.

Next we determine the bounds on the growth of functions in the class $A_o(n, B_k, z_o)$ (cf. (6.2.8)).

Theorem 6.4.2 If $f \in A_o(n, B_k, z_o)$, $B_{k+1} \geq B_k > |z_o| > 0$, $k \geq n+1$, $n = 1, 2, 3, \ldots$, $-1 < z_o < 1$, then,

(6.4.3) $$\operatorname{Max}\left(0,\ f'(0)\left(r - \frac{r^{n+1}}{B_{n+1}}\right)\right) \leq |f(z)| \leq f'(0)\left(r + \frac{r^{n+1}}{B_{n+1}}\right)$$

in the disc U where $|z| = r$. The inequality is sharp for $B_{n+1} \geq 1$.

Proof. The theorem follows from the relation (6.2.9) and Theorem 6.4.1. The function f_{n+1} (cf. (6.2.10)) gives sharpness on the right hand side of (6.4.3) at $z = |z| e^{i\pi/n}$. The same function gives sharpness on the left hand side of (6.4.3) at $z = |z|$ when $B_{n+1} \geq 1$.

Remark. For $f \in S_1(A, B, z_o)$ or $K_1(A, B, z_o)$ (cf. Section 1.4), the upper bound in (6.4.3) is better than that found by Lakshma Reddy and Padmanabhan [68]. The lower bound in (6.4.3) is smaller than that found in [68]. For the sharp function the upper bound in (6.4.3) and that found in [68] are same. Similarly, for the sharp function the lower bound in (6.4.3) and that obtained in [68] are same.

Next, the bounds on distortion of functions in the class $A\{n, M_k\}$ are found.

<u>Theorem</u> 6.4.3 <u>If</u> $f \in A\{n,M_k\}$ <u>and</u> $|M_k/k| \le |M_{k+1}/(k+1)|$, $k \ge n+1$, $n = 1,2,3,\ldots,$ <u>then, for</u> z <u>in the unit disc</u> U,

$$\text{(6.4.4)} \qquad \operatorname{Max}\left(0,\ 1 - \frac{(n+1)r^n}{|M_{n+1}|}\right) \le |f'(z)| \le 1 + \frac{(n+1)r^n}{|M_{n+1}|}$$

<u>where</u> $|z| = r$. <u>The inequality is sharp for</u> $|M_{n+1}| \ge n+1$.

<u>Proof</u>. We have (cf. (6.4.2)), for $z \in U$,

$$f'(z) = 1 + \sum_{k=n+1}^{\infty} k\lambda_k\, z^{k-1}/M_k,$$

where $\lambda_k \ge 0$, $\sum_{k=n+1}^{\infty} \lambda_k \le 1$. Hence,

$$1 - \frac{(n+1)r^n}{|M_{n+1}|} \le |f'(z)| \le 1 + \frac{(n+1)r^n}{|M_{n+1}|}$$

which gives the inequality (6.4.4). Equality in the right hand side in (6.4.4) holds for the function $z + z^{n+1}/M_{n+1}$ at $z = r\, e^{i(\arg M_{n+1})/n}$. Equality holds in the left hand side of (6.4.4) for the same function at $z = r\, e^{i(\arg M_{n+1} + \pi)/n}$ when $|M_{n+1}| \ge n+1$.

<u>Remarks</u>. 1. For $M_{k+1}/(k+1) \le M_k/k < 0$, $k \ge n+1$, Theorem 6.4.4 gives the distortion bounds for functions in the class $A\{n,M_k\} = A(n,\{-M_k\})$ (cf. Section 1.4) found earlier by Sekine [124]

2. For $M_k = k\, e^{i((k-1)\beta-\pi)}/(1-\alpha)$, $\beta \in \mathbb{R}$, Theorem 6.4.4 gives the distortion bounds for functions in the class $\bigcup_{\beta\in\mathbb{R}} A\{n,M_k\} = R_\alpha$ (cf. Section 1.4) found earlier by Srivastava

and Owa [146], where $0 \le \alpha < 1$.

3. The distortion theorems for the classes

$$\bigcup_{\beta\in\mathbb{R}} A\{1, ke^{i((k-1)\beta-\pi)}\} = SV$$

$$\bigcup_{\beta\in\mathbb{R}} A\{1, (k-\alpha)e^{i((k-1)\beta-\pi)}/(1-\alpha)\} = SV^*(\alpha),$$

$0 \le \alpha < 1$, (cf. Section 1.4) found earlier by Silverman [133] follow from Theorem 6.4.5.

Next we determine distortion bounds for functions in the class $A_o(n, B_k, z_o)$ (cf. (6.2.8)).

<u>Theorem</u> 6.4.4 <u>If</u> $f \in A_o(n, B_k, z_o)$, $B_{k+1}/(k+1) \ge B_k/k > |z_o|$, $k \ge n+1$, $-1 < z_o < 1$, <u>then</u>, <u>for</u> z <u>in</u> <u>the</u> <u>unit</u> <u>disc</u> U,

(6.4.5) $$\operatorname{Max}\left(0, f'(0)\left(1-\frac{n+1}{B_{n+1}}r^n\right)\right) \le |f'(z)| \le f'(0)\left(1+\frac{n+1}{B_{n+1}}r^n\right)$$

<u>where</u> $|z| = r$. <u>The</u> <u>inequality</u> <u>is</u> <u>sharp</u> <u>when</u> $B_{n+1} \ge n+1$.

<u>Proof</u>. The relation (6.2.9) and Theorem 6.4.3 give the required inequality. The function f_{n+1} (cf.(6.2.10)) gives sharpness on the right hand side of (6.4.5) at $z = |z|e^{i\pi/n}$. The same function gives sharpness on the left hand side of (6.4.5) at $z = |z|$.

Next, bound on $|f^{(j)}(z)|$ for functions f in the class $A\{n, M_k\}$ is found where $f^{(j)}(z)$ denotes the j-th derivative of f at the point z.

<u>Theorem</u> 6.4.5 <u>If</u> $f \in A\{n, M_k\}$, $|M_k|/k^p \le |M_{k+1}|/(k+1)^p$, $2 \le p \le n+1$, $k \ge n+1$, $n = 1,2,3, \dots,$ <u>then for</u> z <u>in the unit disc</u> U,

$$|f^{(j)}(z)| \le (n+1) \frac{\prod_{i=2}^{j} (n+i)}{|M_{n+1}|} r^{n-j+1} \quad , \; |z| = r, \; 2 \le j \le p.$$

<u>Proof.</u> For $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in A\{n, M_k\}$, we have,

$$\frac{|M_{n+1}|}{(n+1)^m} \sum_{k=n+1}^{\infty} k^m |a_k| \le \sum_{k=n+1}^{\infty} k^p \frac{M_k}{k^p} a_k = \sum_{k=n+1}^{\infty} M_k a_k \le 1$$

for integer m, $1 \le m \le p$ and $2 \le p \le n+1$. Hence

$$\sum_{k=n+1}^{\infty} k^m |a_k| \le \frac{(n+1)^m}{|M_{n+1}|} .$$

For any j with $2 \le j \le p$,

$$z^j f^{(j)}(z) = \sum_{k=n+1}^{\infty} \{\prod_{i=1}^{j} (k+1-i)\} a_k z^k$$

for $z \in U$. Further, using the fact that for $\prod_{i=1}^{j} (k-1+i) = \prod_{i=1}^{j} A_i k^i$, $j \ge 2$, $\sum_{i=1}^{j} A_i (n+1)^{i-1} = \prod_{i=2}^{j} (n+i)$ (cf. Sekine [124]), we have for z in the unit disc U,

$$\begin{aligned} |z^j f^{(j)}(z)| &\le r^{n+1} \sum_{k=n+1}^{\infty} \{\prod_{i=1}^{j} (k-1+i)\} |a_k| \\ &= r^{n+1} \sum_{k=n+1}^{\infty} \{\sum_{i=1}^{j} A_i k^i\} |a_k| \\ &= r^{n+1} \sum_{i=1}^{j} A_i (\sum_{k=n+1}^{\infty} k^i |a_k|) \\ &\le r^{n+1} \sum_{i=1}^{j} A_i \frac{(n+1)^i}{|M_{n+1}|} \end{aligned}$$

$$= \frac{r^{n+1}\,(n+1)}{|M_{n+1}|} \sum_{i=1}^{j} A_i\,(n+1)^{i-1}$$

$$= \frac{r^{n+1}\,(n+1)}{|M_{n+1}|} \prod_{i=2}^{j} (n+i)$$

where $|z| = r$. This gives the required inequality.

<u>Remark</u>. For $M_{k+1} \le M_k < 0$, Theorem 6.4.5 reduces to the analogous result for functions in the class $A\{n,M_k\} = A(n,\{-M_k\})$ (cf. Section 1.4) found earlier by Sekine [124].

The following result gives an upper bound on $|f^{(j)}(z)|$ when f is in $A_o(n,B_k,z_o)$.

<u>Theorem</u> 6.4.6 <u>If</u> $f \in A_o(n,B_k,z_o)$, $B_k/k^p \le B_{k+1}/(k+1)^p$, $k \ge n+1$, $2 \le p \le n+1$, $n = 1,2,3, \ldots,$ <u>then</u>, <u>for</u> z <u>in</u> <u>the</u> <u>unit</u> <u>disc</u> U,

$$|f^{(j)}(z)| \le \frac{f'(0)(n+1)\prod_{i=2}^{j}(n+i)}{B_{n+1}}\, r^{n-j+1}, \quad |z| = r, \quad 2 \le j \le p.$$

<u>Proof</u>. The theorem follows from the relation (6.2.9) and Theorem 6.4.5.

6.5 This section is devoted to find the radii for properties like convexity or starlikeness in the class $A\{n,M_k\}$. The convolution $(f_1*f_2* \ldots * f_p)(z)$, where $f_i \in A\{n,M_k\}$ is also studied in this section.

First we observe the following necessary condition for functions in the classes $T(b,d_2,B_k)$ or $C(b,d_2,B_k)$.

<u>Proposition</u> 6.5.1 If $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n) \in T(b,d_2,B_k)$ (or $C(b,d_2,B_k)$) then $b_1 = b$ and

$$0 < b_2 \le b(b + \frac{d_2}{B_2}). \tag{6.5.1}$$

The inequality is sharp.

<u>Proof</u>. For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n = z/(1+\sum_{n=1}^{\infty} b_n z^n) \in T(b,d_2,B_k)$ (or $C(b,d_2,B_k)$), we have $b_1 = a_2 = b$ and $b_2 = a_2 b_1 + a_3 \le b(b+d_2/B_2)$ which is the assertion of the proposition. The function $z-bz^2 - d_2 bz^3/B_2 \in T(b,d_2,B_k)$, when $B_2 \ge 3d_2 b/(1-2b)$ ($\in C(b,d_2,B_k)$ when $B_2 \ge 9d_2 b/(1-4b)$) gives sharpness in the right hand side of (6.5.1). The left hand side inequality of (6.5.1) is sharp in the sense that no better lower bound exists, which can be seen with the functions $f(z) = z-bz^2 \in T(b,d_2,B_k)$ when $0 < b \le 1/2$ ($\in C(b,d_2,B_k)$ when $0 < b \le 1/4$).

In the following result the largest $r > 0$ such that $\text{Re } f'(z) > \alpha$, $0 \le \alpha < 1$, in the disc $|z| < r$ for all $f \in A\{n,M_k\}$ is found.

<u>Theorem</u> 6.5.1 <u>The radius for</u> $\text{Re } f'(z) > \gamma$, $0 \le \gamma < 1$, <u>in the class</u> $A\{n,M_k\}$, $n = 1,2,3, \ldots$, <u>is</u>

$$r_1 = \inf_{k \ge n+1} \left(\frac{(1-\gamma)|M_k|}{k} \right)^{1/(k-1)}. \tag{6.5.2}$$

<u>Proof</u>. For $f \in A\{n,M_k\}$ and $|z| < r_1$, where r_1 is as in (6.5.2), we get that $\text{Re } f'(z) > \gamma$. Thus, the radius for $\text{Re } f'(z) > \gamma$ in $A\{n,M_k\}$ is at least r_1. The functions $z+z^k/M_k$, $k \ge n+1$ together

give that, in fact, r_1 is the radius for Re $f'(z) > \gamma$ in $A\{n,M_k\}$.

Remark. It follows [60] from Theorem 6.5.1 that functions in $A\{n,M_k\}$ are univalent in the disc $|z| < r_1$.

Theorem 6.5.2 The radius for starlikeness of order γ $(0 \le \gamma < 1)$ in the class $A\{n,M_k\}$, with $|M_k| \ge 1$, $k \ge n+1$, $n = 1,2,3, \ldots,$ is

$$r_2 = \inf_{k\ge n+1} \left(\frac{(1-\gamma)|M_k|}{k-\gamma} \right)^{1/(k-1)}. \tag{6.5.3}$$

Proof. For $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in A\{n,M_k\}$, $|z| < r_2$, where r_2 is as in (6.5.3), we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{z + \sum_{k=n+1}^{\infty} ka_k z^k}{z + \sum_{k=n+1}^{\infty} a_k z^k} - 1 \right|$$

$$= \left| \frac{\sum_{k=n+1}^{\infty} (k-1)a_k z^k}{z + \sum_{k=n+1}^{\infty} a_k z^k} \right|$$

$$\le \frac{\sum_{k=n+1}^{\infty} (k-1)|a_k| r^{k-1}}{1 - \sum_{k=n+1}^{\infty} |a_k| r^{k-1}}$$

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le 1-\gamma \tag{6.5.4}$$

since, for $k \ge n+1$,

$$(k-\gamma)\, r^{k-1} < (1-\gamma)\, |M_k|$$

and

$$\sum_{k=n+1}^{\infty} (k-\gamma)\, |a_k|\, r^{k-1} \le (1-\gamma) \sum_{k=n+1}^{\infty} M_k\, a_k \le 1-\gamma$$

where $|z| = r$. Inequality (6.5.4) gives that f is starlike of order γ in $|z| < r_2$. Thus the radius for starlikeness of order γ in $A\{n,M_k\}$ is at least r_2. The functions $f_k(z) = z+z^k/M_k$, with $|M_k| \geq 1$ and $k \geq n+1$, together give that; in fact, r_2 is the radius for starlikeness of order γ in $A\{n,M_k\}$.

<u>Theorem</u> 6.5.3 <u>The radius for convexity of order</u> γ $(0 \leq \gamma < 1)$ <u>in the class</u> $A\{n,M_k\}$ <u>with</u> $|M_k| \geq k$, $k \geq n+1$, $n = 1,2,3, \ldots,$ <u>is</u>

$$r_3 = \inf_{k\geq n+1} \left(\frac{(1-\gamma)|M_k|}{(k-\gamma)k} \right)^{1/(k-1)} . \tag{6.5.5}$$

<u>Proof</u>. For $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in A\{n,M_k\}$, $\sum_{k=n+1}^{\infty} M_k a_k \leq 1$. If $|z| < r_3$, where r_3 is as in (6.5.5), then $r^{k-1} < (1-\gamma)|M_k|/k(k-\gamma)$ for $k \geq n+1$. Therefore $|zf''(z)/f'(z)| \leq 1-\gamma$ which gives that $\mathrm{Re}\,(1+zf''(z)/f'(z)) > \gamma$. Thus, the radius for convexity of order γ in $A\{n,M_k\}$ is at least r_3. The functions $f_k(z) = z+z^k/M_k$ with $k \geq n+1$, together give that, in fact, r_3 is the radius for convexity of order γ in $A\{n,M_k\}$.

<u>Remarks</u>. 1. By taking $n = 1$, $M_k = e^{i((k-1)\beta-\pi)}/(1-\alpha)$, $k \geq n+1$, $\beta\in\mathbb{R}$, Theorems 6.5.1 and 6.5.2 give the analogous results for the class $\bigcup_{\beta\in\mathbb{R}} A\{n,M_k\} = \mathcal{B}_\alpha$ (cf. Section 1.4) found by Srivastava and Owa [146] where $0 \leq \alpha < 1$.

2. By taking $n = 1$, $M_k = ke^{i((k-1)\beta-\pi)}/(1-\alpha)$, $k \geq n+1$, $\beta \in \mathbb{R}$, Theorem 6.5.3 gives the analogous result for the class $\bigcup_{\beta\in\mathbb{R}} A\{n,M_k\} = R_\alpha$ (cf. Section 1.4) found by Srivastava and Owa

[146] where $0 \le \alpha < 1$.

3. By taking $n = 1$, $M_k = -k(1+\mu\beta)/(1+\mu)\beta(1-\alpha)$, $k \ge 2$, where $0 \le \alpha < 1$, $0 < \beta \le 1$ and $0 \le \mu \le 1$, Theorem 6.5.3 gives the analogous result for the class $A[1,M_k] = P^*(\alpha,\beta,\mu)$ (cf. Section 1.7) found by Owa and Aouf [92].

4. The statements and proofs of Theorems 6.5.1 through 6.5.3 continue to hold for the class $A_o(n,B_k,z_o)$, $n = 1,2,3, \ldots$, $B_k > |z_o| > 0$, $k \ge n+1$, $-1 < z_o < 1$. Thus, the radius for $\mathrm{Re}\, f'(z) > \gamma$, $0 \le \gamma < 1$ in $A_o(n,B_k,z_o)$ is $\inf_{k\ge n+1} ((1-\gamma)B_k/k)^{1/(k-1)}$. The radius for starlikeness of order γ, $0 \le \gamma < 1$, in $A_o(n,B_k,z_o)$, $B_k \ge 1$, $k \ge n+1$, is $\inf_{k\ge n+1} ((1-\gamma)B_k/(k-\gamma))^{1/(k-1)}$. Finally the radius for convexity of order γ, $0 \le \gamma < 1$, in the class $A_o(n,B_k,z_o)$ with $B_k \ge k$ is $\inf_{k\ge n+1} ((1-\gamma)B_k/(k-\gamma)k)^{1/(k-1)}$.

5. Choosing $B_k = (k-\alpha)/(1-\alpha)$, the above Remark 4 gives the radius for convexity in the class $S_o^*(\alpha,z_o)$ (cf. Section 1.4) found earlier by Silverman [131].

6. Choosing $B_k = (k(B+1)-(A+1))/(B-A)$, $-1 \le A < B \le 1$, the above Remark 4 gives the radius for convexity in the class $S_1(A,B,z_o)$ (cf. Section 1.4) found earlier by Lakshma Reddy and Padmanabhan [68].

The result given below is on the convolution of functions in the class $A[n,M_k]$.

<u>Theorem</u> 6.5.4 <u>If</u> $f_i(z) = z + \sum_{k=n+1}^{\infty} a_{k,i} z^k \in A(n,M_k)$ <u>where</u> $i = 1,2,\ldots, p,\ p \geq 2,$ <u>then the Hadamard product</u>

$$(6.5.6) \qquad (f_1 * f_2 * \ldots * f_p)(z) = z + \sum_{k=n+1}^{\infty} \left(\prod_{i=1}^{p} a_{k,i} \right) z^k$$

<u>is in the class</u> $A(n,M_k^p)$. <u>The number</u> M_k^p <u>is the best possible one.</u>

<u>Proof</u>. We prove the theorem for $p = 2$, the proof being similar if $p > 2$. The assumption implies that

$$(6.5.7) \qquad \sum_{k=n+1}^{\infty} M_k a_{k,i} \leq 1$$

for $i = 1,2$. Further,

$$\sum_{k=n+1}^{\infty} M_k^2 a_{k,1} a_{k,2} \leq \sum_{k=n+1}^{\infty} M_k (a_{k,1} a_{k,2})^{1/2}$$

$$\leq \left(\sum_{k=n+1}^{\infty} M_k a_{k,1} \right)^{1/2} \left(\sum_{k=n+1}^{\infty} M_k a_{k,2} \right)^{1/2}$$

by Schwarz inequality. Thus, from the inequality (6.5.7), we have

$$\sum_{k=n+1}^{\infty} M_k^2 a_{k,1} a_{k,2} \leq 1.$$

This and the relation

$$\arg (a_{k,1} a_{k,2}) = - \arg (M_k^2)$$

together give the required assertion. The functions $z+z^k/M_k$, $k \geq n+1$, give sharpness in the sense that the sequence $\{M_k^p\}_{k=n+1}^{\infty}$ can not be replaced by a sequence $\{N_k^p\}_{k=n+1}^{\infty}$ with $|M_k| \leq |N_k|$, $\arg M_k = \arg N_k$, $k \geq n+1$ and at least for one m, $m \geq n+1$, $|M_m| < |N_m|$ such that $(f_1 * \ldots * f_p) \in A(n,N_k^p)$ whenever f_i is in $A(n,M_k)$, $i = 1, \ldots, p$.

Finally, we obtain a result on a special transformation $S_p(z)$ on the class $A\{n,M_k\}$.

<u>Theorem</u> 6.5.5 <u>If</u> $f_i(z) = z + \sum_{k=n+1}^{\infty} a_{k,i}\, z^k \in A\{n,M_k\}, i = 1,2, \ldots,p,$ $p \geq 2$, <u>then the transformation</u>

$$S_p(z) = z + \sum_{k=n+1}^{\infty} \left(\sum_{i=1}^{p} a_{k,i}^2 \right) z^k \tag{6.5.8}$$

<u>is in the class</u> $A\{n,M_k^2/p\}$. <u>The number</u> M_k^2/p <u>is the best possible</u>.

<u>Proof</u>. The proof follows easily and is omitted. The functions $z + z^k/M_k$, $k \geq n+1$, give sharpness in the sense that the sequence $\{M_k^2/p\}_{k=n+1}^{\infty}$ can not be replaced by a sequence $\{N_k\}_{k=n+1}^{\infty}$ where $\arg (M_k^2/p) = \arg N_k$, $|M_k^2/p| \leq |N_k|$ and atleast for one m, $m \geq n+1$, $|M_m^2/p| < |N_m|$ such that $S_p \in A\{n,N_k\}$ whenever f_i is in $A\{n,M_k\}$, $i = 1,\ldots, p$.

CHAPTER VII

ON THE POSITIVITY OF REAL PARTS OF LINEAR COMBINATIONS OF ANALYTIC FUNCTIONS

7.1 Let $\phi = \phi\ (f,\ f',\ f'')$ and $\psi = \psi\ (f,\ f',\ f'')$, where f is in a certain subclass of analytic functions in the unit disc U be such that ϕ, ψ as functions of z are analytic and Re $\phi > 0$ in the domain under consideration. In this chapter the following types of problems are studied.

(i) To find the largest number ρ, $0 < \rho < 1$ such that Re $(\phi + \psi) > 0$ in the disc $|z| < \rho$.

(ii) To find the ranges of scalars λ and μ such that Re $(\lambda\phi + \mu\psi) > 0$ in the unit disc U.

Singh and Paul [143] studied Problems (i), (ii) over the classes CV or $S^*(1/2)$ with special choices of functionals ϕ and ψ. They solved Problem (i) with

$$(7.1.1) \qquad \phi = \frac{f}{z}, \quad \psi = \frac{1}{f'}$$

and $\phi = z^2f''/f$, $\psi = zf'/f$ where f is in the class $S^*(1/2)$ and with $\phi = 1 + zf''/f'$, $\psi = 1/f'$ when f is in the class CV. Further, they investigated Problem (ii) with $\phi = zf'/f, \psi = s_n(z,f)/f$ where, f is in the class $S^*(1/2)$, $s_n(z,f)$ being the n-th partial sum of

$f(z)$ and with

(7.1.2) $$\phi = \frac{f}{zf'}, \quad \psi = \frac{1}{f'}$$

where, f is in the class CV.

In this chapter, we solve Problem (i) with ϕ, ψ as in (7.1.2), when f is in the class CV. This problem is also solved with ϕ, ψ as in (7.1.1), when f is in the class S^*. We study Problem (ii) with

$$\phi = \frac{zf'}{f}, \quad \psi = \frac{V_n(z,f)}{f}$$

when $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is in the class $S^*(1/2)$, $V_n(z,f)$ being the de la Vallee Poussin mean of order n of $f(z)$, $n \geq 1$ i.e.,

(7.1.3) $$V_n(z,f) = \frac{n}{n+1} a_1 z + \frac{n(n-1)}{(n+1)(n+2)} a_2 z^2 + \ldots + \frac{n(n-1)(n-2)..2.1}{(n+1)(n+2)...(2n)} a_n z^n .$$

In Section 7.2, Problems (i) and (ii) are studied for functions f in the classes CV, $S^*(1/2)$ or S^*. Sufficient conditions regarding Problem (ii) are determined in Section 7.3. The results in this section give those of Reade et al. [109] and Ahuja and Jain [2] as special cases. The results in the final Section 7.4 deal with Problems (i) and (ii) for functions in the class R_α, of prestarlike functions of order α (cf. Section 1.3) with certain special choices of functionals ϕ and ψ.

7.2 This section consists of results related to Problems (i) and (ii) stated in Section 7.1, when f is in the classes CV, $S^*(1/2)$

or S^*. We need the following result of Ruscheweyh and Sheil-Small [118] in the sequel:

If $f \in CV$, $g \in S^*$ and $F \in \mathcal{A}$ with $\text{Re } F(z) > 0$ in the unit disc U then, for $z \in U$,

$$\frac{f(z)*(g(z)F(z))}{f(z)*g(z)} \in Co(F(U)) \tag{7.2.1}$$

where, $Co(F(U))$ is the convex hull of $F(U)$.
The relation (7.2.1) continues to hold when the functions f, g are in the class $S^*(1/2)$.

The following result gives solution to a problem of the type (ii) for functions $f \in CV$. The fact that $\text{Re } f'(z) > 0$ only in the disc $|z| < 1/\sqrt{2} \cong 0.707$ underlines the significance of this result.

<u>Theorem</u> 7.2.1 <u>If</u> $f \in CV$, <u>then</u>

$$\text{Re}\left[\frac{f(z)}{zf'(z)} + \frac{1}{f'(z)}\right] > 0 \tag{7.2.2}$$

<u>in the disc</u> $|z| < \rho = \sqrt{7/8} \cong 0.93$. <u>The number</u> ρ <u>is the best possible one</u>.

<u>Proof</u>. Let $h(z) = (1-z) + (1-z)^2$. First it is observed that $\text{Re } h(z) > 0$ in the disc $|z| < \rho = \sqrt{7/8}$. We have

$$\text{Re } h(z) = 2-r^2 - 3ru + 2r^2u^2 \equiv \varphi(u)$$

where $z = re^{i\theta}$ and $u = \cos\theta$.

For $3/4 \leq r < \sqrt{7/8}$, we have

$$\partial\varphi/\partial u = 0 \text{ and } \partial^2\varphi/\partial u^2 > 0$$

at $u = 3/4r$. Hence,

$$\min \varphi(u) = \varphi\left(\frac{3}{4r}\right) = \frac{7-8r^2}{8} > 0 .$$

For $r < 3/4$, we have

$$\min \varphi(u) = \varphi(1) = (2-r)(1-r) > 0.$$

Thus, we have shown that

$$\text{(7.2.3)} \qquad \operatorname{Re} h(z) > 0$$

in the disc $|z| < \rho$.

Now by taking $g(z) = z/(1-\rho z)^2$ and $F(z) = \rho h(\rho z)$ in the relation (7.2.1), we have that

$$G(z) \equiv \frac{f(z)*[\rho z(1-\rho z)^{-2} [(1-\rho z) + (1-\rho z)^2]]}{f(z)*[z(1-\rho z)^{-2}]}$$

$$= \rho \left[\frac{f(\rho z)}{\rho z f'(\rho z)} + \frac{1}{f'(\rho z)} \right]$$

takes values in the convex hull of $F(U)$. By the inequality (7.2.3), we get that $\operatorname{Re} F(z) > 0$ in the unit disc U so that $\operatorname{Re} G(z) > 0$ in U and the proof of the theorem is complete. The function $l(z) = z/(1-z)$ shows that the number ρ is the best possible one.

Singh and Singh [142] found that for $f \in S^*(1/2)$, we have $\operatorname{Re} (f(z)/V_n(z,f)) > 0$ in the unit disc U, where $V_n(z,f)$ is the de la Vallee Poussin mean of order n of $f(z)$ given by (7.1.3). This observation is a motivation for our next result.

Theorem 7.2.2 Let $f \in S^*(1/2)$ and

$$(7.2.4)\qquad L(z) = \operatorname{Re}\left[\lambda \frac{zf'(z)}{f(z)} + \mu \frac{V_n(z,f)}{f(z)}\right], \qquad z \in U,\ n \geq 1.$$

Then, $L(z) > 0$ in the unit disc U, if (i) $\lambda \geq 0$, $\mu \geq 0$ and at least one of them is nonzero, or (ii) μ is a complex number and $\lambda > 4n\,|\mu|/(n+1)$. The result is sharp for $n = 1$, in the sense that the ranges of λ and μ can not be increased.

Proof. First it is shown that the function

$$h(z) = \mu\left[\frac{3n}{(n+1)(n+2)}(1-z) + \frac{5n(n-1)}{(n+1)(n+2)(n+3)}(1-z^2) + \dots + \frac{(2n+1)n(n-1)(n-2)\dots 2.1}{(n+1)(n+2)\dots(2n)(2n+1)}(1-z^n)\right] + \frac{\lambda}{1-z}$$

has $\operatorname{Re} h(z) > 0$ in the unit disc U. In fact,

$$\operatorname{Re} h(z) \geq -|\mu|\left[\frac{3n}{(n+1)(n+2)}(1+r) + \frac{5n(n-1)}{(n+1)(n+2)(n+3)}(1+r^2) + \dots + \frac{(2n+1)n(n-1)(n-2)\dots 2.1}{(n+1)(n+2)\dots(2n)(2n+1)}(1+r^n)\right] + \frac{\lambda}{1+r}$$

$$\geq \frac{\lambda}{2} - \frac{2n}{n+1}|\mu| > 0$$

in the unit disc U where $|z| = r$. Since $f \in S^*(1/2)$, by taking $F(z) = h(z)$ and $g(z) = z/(1-z)$ in the relation (7.2.1), we obtain that $\operatorname{Re} F(z) > 0$ in the unit disc U, which proves the theorem under condition (ii). The proof of the theorem under condition (i) is easy and is omitted. The function $l(z) = z/(1-z)$ shows that for $n = 1$, the ranges of λ and μ can not be increased in the theorem.

For a function $f \in S^*$, we have $\operatorname{Re}(f(z)/z) > 0$, only in the

disc $|z| < 1/\sqrt{2} \cong 0.707$. By the rotation theorem [35] for the class S^*, we have $\mathrm{Re}\, f'(z) > 0$ only in the disc $|z| < \sin\ (\pi/8) \cong 0.38628$. These observations underline the significance of the following result.

<u>Theorem</u> 7.2.3 <u>If</u> $f \in S^*$, <u>then</u>

$$(7.2.5) \qquad \mathrm{Re}\left[\frac{f(z)}{z} + f'(z)\right] > 0$$

<u>in the disc</u> $|z| < \rho = 1/2$. <u>The number</u> ρ <u>is the best possible one</u>.

<u>Proof</u>. For a fixed $z \in U$, the functional

$$L(f) = \frac{f(z)}{z} + f'(z)$$

is a continuous linear functional on $\mathcal{A}$ in the topology of uniform convergence on compact subsets of U. And the set of extreme points of the closed convex hull of S^*, $\overline{Co}(S^*)$ is given by (1.7.1),

$$\mathrm{Ext}\ \{\ \overline{Co}(S^*)\ \} = \{\ \frac{z}{(1-xz)^2} : |x| = 1\ \}.$$

Inequality (7.2.5) being rotation invariant, it is enough to prove the inequality for the Koebe function, $k(z) = z/(1-z)^2$, in view of the Krein-Mil'man theorem (cf. Section 1.7). We have, for $f(z) = k(z)$,

$$\mathrm{Re}\left[\frac{f(z)}{z} + f'(z)\right] = \mathrm{Re}\left[\frac{1}{(1-z)^2} + \frac{1+z}{(1-z)^3}\right] = \frac{2B(u)}{|1-z|^6}$$

for

$$B(u) \equiv 1-3r^2 - 3r(r^2-1)u + 6r^2u^2 - 4r^3u^3$$

in the unit disc U where $z = re^{i\theta}$ and $u = \cos\theta$.

For $r < 1/3$, the function $B(u)$ is decreasing in u. Hence

$$B(u) \geq B(1) = (1-r)^3 > 0.$$

For $r \geq 1/3$,

$$\frac{\partial B}{\partial u}(u_1) = 0 \text{ and } \frac{\partial^2 B}{\partial u^2}(u_1) > 0$$

where, $u_1 = (r-1)/2r$. Hence,

$$B(u) \geq B(u_1) = \frac{1}{2}(1-3r^2-2r^3)$$

$$= -(1+r)^2(r-\rho) > 0$$

for $1/3 \leq r < \rho = 1/2$. Thus, the theorem is proved. It is clear from the proof that the radius ρ is the best possible one.

7.3 This section is devoted to sufficient conditions regarding Problem (ii) stated in Section 7.1.

It may be recalled that a function f in the class A_1 of normalized (cf. Section 1.1) functions, analytic in the unit disc U, with $f(z) \neq 0$ in $U\backslash\{0\}$, may be expressed as $f(z) = z/g(z)$ in U where, $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \in A$. In the following theorem a sufficient condition is determined in terms of the coefficient sequence $\{b_n\}_{n=1}^{\infty}$ for the functional $\operatorname{Re}(\lambda\, zf'(z)/f(z) + \mu\, f(z)/z)$ to be positive in the unit disc U. Here, we allow $\operatorname{Re}(zf'(z)/f(z))$ to take negative values in U.

<u>Theorem</u> 7.3.1 <u>Let</u> $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n) \in A_1$ <u>with</u> b_n<u>'s satisfying</u>

$$(7.3.1) \qquad \sum_{n=1}^{\infty} \{|\lambda(1-n)-1| + |\lambda(1-n)+1|\}\, |b_n| \leq |\lambda+\mu+1| - |\lambda+\mu-1|.$$

for λ, μ *in* $\mathbb{C}$ *and at least one of them is nonzero. Then*

$$\text{(7.3.2)} \qquad \mathrm{Re}\left[\lambda \frac{zf'(z)}{f(z)} + \mu \frac{f(z)}{z}\right] > 0$$

in the unit disc U.

Proof. For $f(z) = z/g(z)$ where $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, we have,

$$\left|\frac{\lambda \frac{zf'(z)}{f(z)} + \mu \frac{f(z)}{z} - 1}{\lambda \frac{zf'(z)}{f(z)} + \mu \frac{f(z)}{z} + 1}\right| = \left|\frac{\lambda \frac{g(z)-zg'(z)}{g(z)} + \frac{\mu}{g(z)} - 1}{\lambda \frac{g(z)-zg'(z)}{g(z)} + \frac{\mu}{g(z)} + 1}\right|$$

$$= \left|\frac{\lambda(g(z)-zg'(z))+\mu-g(z)}{\lambda(g(z)-zg'(z))+\mu+g(z)}\right|$$

$$= \left|\frac{\lambda+\mu-1+\sum_{n=1}^{\infty}(\lambda(1-n)-1)b_n z^n}{\lambda+\mu+1+\sum_{n=1}^{\infty}(\lambda(1-n)+1)b_n z^n}\right|$$

$$\leq \frac{|\lambda+\mu-1|+\sum_{n=1}^{\infty}|\lambda(1-n)-1|\,|b_n|}{|\lambda+\mu+1|-\sum_{n=1}^{\infty}|\lambda(1-n)+1|\,|b_n|}$$

Thus,

$$\text{(7.3.3)} \qquad \left|\frac{\lambda \frac{zf'(z)}{f(z)} + \mu \frac{f(z)}{z} - 1}{\lambda \frac{zf'(z)}{f(z)} + \mu \frac{f(z)}{z} + 1}\right| \leq 1$$

by the condition (7.3.1). Now the inequality (7.3.3) gives the inequality (7.3.2).

Taking $\lambda = 1$ and $\mu = 0$ in Theorem 7.3.1, the following corollary is obtained.

<u>Corollary</u> 7.3.1 (Reade et al. [109]) If $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n) \in A_1$ and

(7.3.4) $$|b_1| + \sum_{n=2}^{\infty} (n-1)\, |b_n| \le 1$$

then $f \in S^*$.

Taking $\lambda = 0$ and $\mu = 1$ in Theorem 7.3.1, the following corollary is obtained, which is same as the assertion of Part (i), Corollary 5.4.3, for $\alpha = 0$.

<u>Corollary</u> 7.3.2 For $f(z) = z/(1 + \sum_{n=1}^{\infty} b_n z^n) \in A_1$, if

$$\sum_{n=1}^{\infty} |b_n| \le 1$$

then $\mathrm{Re}\,(f(z)/z) > 0$ in U.

Next a sufficient condition is determined for the functional $\mathrm{Re}\,(\lambda\,(1+zf''(z)/f'(z)) + \mu\, zf'(z)/f(z))$ to be positive in the unit disc U. Here, we allow $\mathrm{Re}\,(1+zf''(z)/f'(z))$ to take negative values in U.

<u>Theorem</u> 7.3.2 <u>Let</u> $f(z) = z/(1+ \sum_{n=1}^{\infty} b_n z^n) \in A_1$ <u>with</u> b_n's <u>satisfying</u>

(7.3.5) $$(|\lambda|+|2\lambda+\mu|+\mathrm{Re}\,(\lambda+\mu))|b_1| + \sum_{n=2}^{\infty} ((|\lambda|+|2\lambda+\mu|)n+\mathrm{Re}(\lambda+\mu))(n-1)|b_n| \le \mathrm{Re}(\lambda+\mu)$$

<u>where</u> λ, μ <u>are in</u> $\mathbb{C}$ <u>and at least one of them is nonzero. Then</u>

(7.3.6) $$\mathrm{Re}\left[\lambda\,\left(1+ \frac{zf''(z)}{f'(z)}\right) + \mu\, \frac{zf'(z)}{f(z)}\right] > 0, \qquad z \in U.$$

Proof. For $f(z) = z/g(z)$ where $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{A}$, we have

$$(7.3.7) \qquad \lambda \left(1+ \frac{zf''(z)}{f'(z)}\right) + \mu \frac{zf'(z)}{f(z)}$$

$$= \lambda+\mu - \frac{(2\lambda+\mu) \sum_{n=1}^{\infty} nb_n z^n}{1+ \sum_{n=1}^{\infty} b_n z^n} + \frac{\lambda \sum_{n=2}^{\infty} n(n-1)\, b_n z^n}{1+ \sum_{n=2}^{\infty} (1-n)\, b_n z^n}$$

in the unit disc U. For $a = |2\lambda+\mu| \operatorname{Re}(\lambda+\mu)/(|2\lambda+\mu|+|\lambda|)$, we have,

$$(7.3.8) \qquad \left| \frac{(2\lambda+\mu) \sum_{n=1}^{\infty} nb_n z^n}{1+ \sum_{n=1}^{\infty} b_n z^n} \right| \le \frac{|2\lambda+\mu| \sum_{n=1}^{\infty} n|b_n|}{1- \sum_{n=1}^{\infty} |b_n|} \le a,$$

and

$$(7.3.9) \qquad \left| \frac{\lambda \sum_{n=2}^{\infty} n(n-1)\, b_n z^n}{1+ \sum_{n=2}^{\infty} (1-n)\, b_n z^n} \right| \le \frac{|\lambda| \sum_{n=2}^{\infty} n(n-1)|b_n|}{1- \sum_{n=2}^{\infty} (n-1)|b_n|}$$

$$\le \operatorname{Re}(\lambda+\mu) - a$$

by the condition (7.3.5). By using the inequalities (7.3.8) and (7.3.9) in the equation (7.3.7), the inequality (7.3.6) is obtained.

Corollary 7.3.3 (Reade et al. [109]) If $f(z) = z/(1+ \sum_{n=1}^{\infty} b_n z^n) \in \mathcal{A}_1$ with the b_n's satisfying $4|b_1| + \sum_{n=2}^{\infty} (n-1)(3n+1)\, |b_n| \le 1$, then f belongs to CV.

Proof. By taking $\lambda = 1$ and $\mu = 0$ in Theorem 7.3.2, the corollary is obtained.

Corollary 7.3.4 If $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n) \in \mathcal{A}_1$ with the b_n's satisfying

$$(7.3.10) \quad (1+|\lambda|+|1+\lambda|)|b_1| + \sum_{n=2}^{\infty} (1+n(|\lambda|+|1+\lambda|))(n-1)|b_n| \le 1$$

for $\lambda \in \mathbb{C}$, then f is λ-convex in the unit disc U.

Proof. Choosing $\mu = 1 - \lambda$, the condition (7.3.5) becomes the condition (7.3.10). Thus Theorem 7.3.2 gives that the given function f is λ-convex in U.

Corollary 7.3.5 If $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n) \in \mathcal{A}_1$ with the b_n's satisfying

$$(\cos\lambda+\alpha+|e^{i\lambda}+\alpha|)\ |b_1| + \sum_{n=2}^{\infty} ((\alpha+|e^{i\lambda}+\alpha|)n + \cos\lambda)(n-1)|b_n|$$
$$\le \cos\lambda,$$

for $|\lambda| < \pi/2$, $\lambda \in \mathbb{R}$, $0 \le \alpha$, then f is α-λ spiral of order 0.

Proof. Choosing $\lambda = \alpha$, $e^{i\lambda} - \alpha$ in place of μ, Theorem 7.3.2 gives the corollary.

For $\mu = 0$, $\lambda = e^{i\beta}$, $-\pi/2 < \beta < \pi/2$, Theorem 7.3.2 gives the following corollary.

Corollary 7.3.6 (Ahuja and Jain [2]) If $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n) \in \mathcal{A}_1$, $-\pi/2 < \beta < \pi/2$ and

$$(3 + \cos\beta)\ |b_1| + \sum_{n=2}^{\infty} (3n + \cos\beta)(n-1)\ |b_n| \le \cos\beta$$

then f is a β-Robertson function of order 0 (cf. Definition 1.6.1) in U.

7.4 The aim of this section is to study Problems (i) and (ii) stated in Section 7.1 for functions in the class R_α (cf. Section 1.3) of prestarlike functions of order α, $\alpha \le 1$.

A function f analytic in the unit disc U, normalized by $f(0) = 0$, $f'(0) \neq 0$ is said to be prestarlike of order α in U, $\alpha \le 1$, if

$$\begin{cases} \operatorname{Re} \dfrac{f(z)}{zf'(0)} > \dfrac{1}{2}, & z \in U,\ \alpha = 1, \\ \\ \dfrac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S_\alpha, & \alpha < 1 \end{cases}$$

where, S_α is as in Section 1.3. The class of prestarlike functions of order α in U is denoted by R_α.

For $\alpha \le 1$, $f \in R_\alpha$ and $p \in S_\alpha$, Ruscheweyh [116] found that, if F(z) is analytic with positive real part in the unit disc U, then

$$\operatorname{Re} \frac{f(z)*(p(z)F(z))}{f(z)*p(z)} > 0 \qquad \text{in } U. \tag{7.4.1}$$

Throughout in the sequel, for a function f analytic in U, by $\mathbb{D}^\gamma f(z)$, $\gamma \ge -1$, we mean that for $z \in U$,

$$\mathbb{D}^\gamma f(z) = \frac{z}{(1-z)^{\gamma+1}} * f(z) .$$

For a function f belonging to $R_{(1-\gamma)/2}$ ($\gamma \ge 0$) we have, $\operatorname{Re}\,((\mathbb{D}^{\gamma+1} f(z))/\mathbb{D}^\gamma f(z)) > 1/2$ in U. Since $R_\alpha \subseteq R_\eta$ for $\alpha \le \eta \le 1$, [116], we have that the function f is also in the class

$R_{(1-\beta)/2}$ where $\beta = \gamma - 1$, which is equivalent to

$$\mathrm{Re}\ ((\mathbb{D}^{\gamma} f(z))/\mathbb{D}^{\gamma-1} f(z)) > 1/2$$

in the unit disc U. These observations make the following result significant.

<u>Theorem</u> 7.4.1 <u>For</u> $f \in R_{(1-\gamma)/2}$, <u>let</u>

$$(7.4.2) \qquad L(z) = \mathrm{Re}\left[\lambda \frac{\mathbb{D}^{\gamma+1} f(z)}{\mathbb{D}^{\gamma} f(z)} + \mu \frac{\mathbb{D}^{\gamma-1} f(z)}{\mathbb{D}^{\gamma} f(z)}\right], \qquad z \in U.$$

<u>where</u> $\gamma \geq 0$. <u>Then</u>, $L(z) > 0$ <u>in the unit disc U if</u> (i) $\lambda \geq 0, \mu \geq 0$, <u>and at least one of</u> λ, μ <u>is nonzero</u>, <u>or</u> (ii) $\lambda > 4|\mu|$ <u>when</u> μ <u>is a complex number.</u> <u>The result is sharp in the sense that the ranges of</u> λ <u>and</u> μ <u>can not be increased</u>.

<u>Proof</u>. Let $\lambda > 4|\mu|$ where $\mu \in \mathbb{C}$. First it is shown that the function

$$h(z) = \frac{\lambda}{1-z} + \mu\ (1-z)$$

has $\mathrm{Re}\ h(z) > 0$ in the unit disc U. We have

$$\mathrm{Re}\ h(z) \geq \frac{\lambda}{1+r} - |\mu|(1+r)$$
$$> \frac{\lambda}{2} - 2|\mu| > 0$$

in the unit disc U where $|z| = r$. Now by taking $F(z) = h(z)$ and $p(z) = z/(1-z)^{\gamma+1}$ in the inequality (7.4.1), we obtain part (ii) of the theorem. The proof of part (i) is straight forward and is omitted. The function $\ell(z) = z/(1-z) \in R_{(1-\gamma)/2}$ ($\gamma \geq 0$) gives sharpness of the scalars λ and μ in the theorem.

The following result is important in view of the fact that $\operatorname{Re}((\mathbb{D}^{\gamma-2} l(z))/\mathbb{D}^{\gamma} l(z)) > 0$ only in the disc $|z| < 1/\sqrt{2}$, where $l(z) = z/(1-z)$ and $\gamma \geq 1$.

<u>Theorem</u> 7.4.2 <u>If</u> $f \in R_{(1-\gamma)/2}$, $\gamma \geq 1$, <u>then for all</u> λ <u>and</u> μ <u>such that</u> $\lambda > 2\mu \geq 0$, <u>we have for</u> $z \in U$,

$$\operatorname{Re}\left[\lambda \frac{\mathbb{D}^{\gamma-1} f(z)}{\mathbb{D}^{\gamma} f(z)} + \mu \frac{\mathbb{D}^{\gamma-2} f(z)}{\mathbb{D}^{\gamma} f(z)}\right] > 0 \tag{7.4.3}$$

<u>Proof</u>. First it is shown that the function

$$h(z) = \lambda(1-z) + \mu(1-z)^2$$

has $\operatorname{Re} h(z) > 0$ in the unit disc U. We have

$$\begin{aligned}\operatorname{Re} h(z) &= \lambda \operatorname{Re}(1-z) + \mu \operatorname{Re}(1-z)^2 \\ &= (\lambda-2\mu)(1-r\cos\theta) + \mu((1-r^2) + 2(1-r\cos\theta)^2) \\ &> 0\end{aligned}$$

in U where $z = re^{i\theta}$.

Now by taking $F(z) = h(z)$ and $p(z) = z/(1-z)^{\gamma+1}$ in the inequality (7.4.1), the required inequality (7.4.3) is obtained.

The property that $\operatorname{Re}((\mathbb{D}^{\gamma+2} l(z))/\mathbb{D}^{\gamma} l(z)) > 0$ only in the disc $|z| < 1/\sqrt{2}$, motivates us to obtain the following result, where $l(z) = z/(1-z)$ and $\gamma \geq -1$.

<u>Theorem</u> 7.4.3 <u>If</u> $f \in R_{(1-\gamma)/2}$, <u>with</u> $\gamma \geq -1$, <u>then</u>

$$\operatorname{Re}\left[\frac{\mathbb{D}^{\gamma+1} f(z)}{\mathbb{D}^{\gamma} f(z)} + \frac{\mathbb{D}^{\gamma+2} f(z)}{\mathbb{D}^{\gamma} f(z)}\right] > 0 \tag{7.4.4}$$

<u>in the disc</u> $|z| < \rho = \sqrt{4\sqrt{2}-5} \cong 0.81$. <u>The number</u> ρ <u>is the best possible one</u>.

<u>Proof</u>. First it is shown that the function

$$h(z) = \frac{1}{1-z} + \frac{1}{(1-z)^2}$$

has Re $h(z) > 0$ in the disc $|z| < \rho = \sqrt{4\sqrt{2} - 5}$. For $1/(1-z) = R\, e^{i\zeta}$, we have

$$\frac{1}{1+r} \le R \le \frac{1}{1-r}$$

and

$$\cos\zeta = \frac{1+R^2-r^2R^2}{2R}$$

where $|z| = r$ in the unit disc U. Thus,

$$2\mathrm{Re}\, h(z) = 2(R\cos\zeta + R^2\cos 2\zeta)$$

$$= 2+(1-3r^2)t + (1-r^2)^2\, t^2 \equiv \varphi(t)$$

where $t = R^2$. For $r \ge \sqrt{7}-2$ we have $t_1 = (3r^2-1)/2(1-r^2)^2$ is in the range of t,

$$\frac{\partial\varphi}{\partial t}(t_1) = 0 \text{ and } \frac{\partial^2\varphi}{\partial t^2}(t_1) > 0 .$$

Hence, for $r \ge \sqrt{7} - 2$,

$$\min \varphi(t) = \varphi(t_1) = \frac{8(1-r^2)^2-(3r^2-1)^2}{4(1-r^2)^2} > 0$$

when $r < \rho$.

For $r < \sqrt{7} - 2$, we have

$$\min \varphi(t) = \varphi\left(\frac{1}{(1+r)^2}\right) = \frac{2(2+r)}{(1+r)^2} > 0.$$

Thus, we have Re $h(z) > 0$ in the disc $|z| < \rho$. Now by taking $F(z) = h(\rho z)$, $p(z) = \rho z/(1-\rho z)^{\gamma+1}$ and $f(\rho z)$ in place of $f(z)$ in the inequality (7.4.1), we have that the function

$$\frac{f(\rho z) * \left[\frac{\rho z}{(1-\rho z)^{\gamma+1}}\left(\frac{1}{1-\rho z}+\frac{1}{(1-\rho z)^2}\right)\right]}{f(\rho z) * \frac{\rho z}{(1-\rho z)^{\gamma+1}}} = \frac{\mathbb{D}^{\gamma+1}f(\rho z)+\mathbb{D}^{\gamma+2}f(\rho z)}{\mathbb{D}^{\gamma}f(\rho z)}$$

takes values in the convex hull of $F(U)$ from which the required inequality follows. The function $l(z) = z/(1-z) \in R_{(1-\gamma)/2}$ gives sharpness of the radius ρ.

<u>Remark</u>. For $\gamma = -1$, it follows from Theorem 7.4.3 that the radius for

$$(7.4.5) \qquad \mathrm{Re}\left[\frac{f(z)}{z} + f'(z)\right] > 0$$

in the class R_1 is $\sqrt{4\sqrt{2}-5}$. Bhooshnurmath and Swamy [10] studied the class consisting of functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in A_1$ which satisfy (7.4.5) in the unit disc U with $a_n \geq 0$, $n \geq 2$.

The following result is also concerning Problem (i) stated in Section 7.1.

<u>Theorem</u> 7.4.4 <u>If</u> $f \in R_{(1-\gamma)/2}$, $\gamma \geq 1$, <u>then</u>

$$(7.4.6) \qquad \mathrm{Re}\left[\frac{\mathbb{D}^{\gamma-1}f(z)}{\mathbb{D}^{\gamma}f(z)} + \frac{\mathbb{D}^{\gamma-2}f(z)}{\mathbb{D}^{\gamma}f(z)}\right] > 0$$

<u>in the disc</u> $|z| < \rho = \sqrt{7/8} \cong 0.93$. <u>The number</u> ρ <u>is the best possible one</u>.

<u>Proof</u>. In the proof of Theorem 7.2.1, it has been seen that the function

$$h(z) = (1-z) + (1-z)^2$$

has Re $h(z) > 0$ only in the disc $|z| < \rho = \sqrt{7/8}$. Now, by taking $F(z) = h(\rho z)$, $p(z) = \rho z/(1-\rho z)^{\gamma+1}$ and $f(\rho z)$ in place of $f(z)$ in the inequality (7.4.1), we have that the function

$$\frac{f(\rho z) * \left[\frac{\rho z}{(1-\rho z)^{\gamma+1}}((1-\rho z)+(1-\rho z)^2)\right]}{f(\rho z) * \frac{\rho z}{(1-\rho z)^{\gamma+1}}} = \frac{\mathbb{D}^{\gamma-1} f(\rho z) + \mathbb{D}^{\gamma-2} f(\rho z)}{\mathbb{D}^{\gamma} f(\rho z)}$$

takes values in the convex hull of $F(U)$ from which the required inequality follows. The function $l(z) = z/(1-z) \in R_{(1-\gamma)/2}$ gives sharpness of the radius ρ.

Finally we close the section with the following observation on the class $CV(\alpha)$, $0 < \alpha < 1$:

If $f \in CV(\alpha)$, $0 < \alpha < 1$, then,

$$\mathrm{Re}\left[\ 1 + \frac{zf''(z)}{(1-\alpha)f'(z)} + \frac{1}{(f'(z))^{1/(1-\alpha)}}\right] > 0$$

in the disc $|z| < \rho = (\sqrt{5}-1)/\sqrt{2} \cong 0.874$. The radius ρ can not be replaced by any larger one. This inequality follows from the inequality (1.3.8) and the relation that for $f \in CV(\alpha)$, the function $\int_0^z (f'(\zeta))^{1/(1-\alpha)}\, d\zeta \in CV$ (cf. (3.2.1)).

BIBLIOGRAPHY

1. Ahuja, O.P.: The Bieberbach conjecture and its impact on the developments in Geometric Function theory, Math. Chronicle **15** (1986), 1-28.

2. Ahuja, O.P.; Jain, P.K.: On the spiral-likeness of rational functions, Rend. Circ.Mat. Palermo (2) **35** (1986), 376-385 (1987).

3. Ahuja, O.P.; Silverman, H. : Extreme points of families of univalent functions with fixed second coefficient, Colloq. Math. **54** (1987), 127-137.

4. Alexander, J.W.: Functions which map the interior of the unit circle upon simple regions, Ann. of Math. **17** (1915), 12-22.

5. Anh, V.V.; Tuan, P.D.: Extremal problems for a class of functions of positive real part and applications, J. Austral. Math. Soc. Ser. A **41** (1986), 152-164.

6. Bajpai, S.K.; Silvia, E.M.: On certain classes of analytic functions, Notices AMS (20) **1** (1973), A-107.

7. Barnard, R.W.: Open problems and conjectures in complex analysis, Computational methods and function theory, Proc., Valporaiso 1989, Lecture notes in Mathematics **1435**, pp. 1-26, Springer Verlag Berlin Heidelberg 1990.

8. Bazilevic, I.E.: On distortion theorems in the theory of univalent functions, Mat. Sb. N.S. **28** (1951), 283-292 (Russian).

9. Bernardi, S.D.: Bibliography of schlicht functions, Mariner Publishing Company, Inc. Tampa, Florida (1982).

10. Bhoosnurmath, S.S. and Swamy, S.R.: Certain classes of analytic functions with negative coefficients, Indian J. Math. **27** (1985), 89-98.

11. Bieberbach, L. : Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, S.-B. Preuss. Akad. Wiss.,1916, 940-955.

12. __________ : Lehrbuch der Funktionen theorie, Bd. II, Zweite Auflage, Leipzig und Berlin, 1931.

13. Bielecki, A. and Lewandowski, Z.: Sur une generalisation de quelques theoremes de M. Biernacki sur les fonctions analytiques, Ann. Polon. Math. **12** (1962) 65-70.

14. Biernacki, M.: Sur quelques majorantes de la theorie des fonctions univalentes, C.R.Acad. Sci. Paris **201** (1935), 256-258.

15. __________ : Sur les fonctions univalentes, Mathematica **12** (1936), 49-64.

16. Brannan, D.A.: Coefficient Regions for univalent polynomials of small degree, Mathematika **14** (1967), 165-169.

17. Brannan, D.A. and Brickman, L.: Coefficient regions for starlike polynomials, Ann. Univ. Mariae Curie - Sklodowska Sect. A, **29** (1975),15-21.

18. Brannan, D.A.; Hayman, W.K.: Research problems in complex analysis, Bull. London Math. Soc. **21** (1989), 1-35.

19. Brickman, L.; Mac Gregor, T.H. and Wilken, D.R.: Convex hulls of some classical families of univalent functions, Trans. Amer. Math. Soc. **156** (1971), 91-107.

20. Brickman, L.; Hallenbeck, D.J.; Mac Gregor, T.H., Wilken, D.R.: Convex hulls and extreme points of families of starlike and convex mappings, Trans. Amer. Math. Soc. **185** (1973), 413-428.

21. Brown, Johnny E: Images of disks under convex and starlike functions, Math. Z. **202** (1989), 457-462.

22. Chen, M.P.: On functions satisfying $Re(f(z)/z) > \alpha$, Tamkang J. Math. **5** (1974), 231-234.

23. Cowling, V.F. and Royster, W.C.: Domains of variability for univalent polynomials, Proc. Amer. Math. Soc. **19** (1968), 767-772.

24. De Branges, Louis: A proof of Bieberbach conjecture, Acta. Math. **154** (1985), 137-152.

25. Deeb, W.: Extreme and support points of families of univalent functions with real coefficients, Math. Rep. Toyama Univ. **8** (1985), 103-111.

26. Duren, P.L., Univalent functions, Springer-Verlag, New York, 1983.

27. Eenigenburg, P.: On the radius of curvature for convex analytic functions, Canad. J. Math. **22** (1970), 486-491.

28. Eenigenburg, P.J.: A class of starlike mappings of the unit disc, Compositio Math. **24** (1972), 235-238.

29. Ezrohi, T.G.: Certain estimates in special classes of univalent functions in the unit circle $|z| < 1$ (Ukrainian), Dopovidi Akad. Nauk Ukrain. RSR (1965), 984-988.

30. Fitzgerald, C.H.; Pommerenke, Ch.: The de Branges theorem on univalent functions, Trans. Amer. Math. Soc. **290** (1985), 683-690.

31. Ganeshan, M.S.: Coefficient regions for polynomials starlike of order α, Int. Conference on new trends in Geom. Function Theory, July 26-29, 1990, Madras Univ. Madras, India.

32. Gelfond, A.O. and Leontev, A.F.: On a generalization of Fourier series, Mat. Sb. (N.S.), **29** (71), (1951), 477-500.

33. Goel, R.M.: Functions starlike and convex of order α, J. London Math. Soc. (2) **9** (1974/75), 128-130.

34. Goel, R.M.; Mehrok, B.S.: On the coefficients of a subclass of starlike functions, Indian J. Pure Appl. Math. **12** (1981), 634-647.

35. Goluzin, G.: On distortion theorems of schlicht conformal mappings (Russian), Rec. Math. Moscow (2), **1** (1936), 127-135.

36. ________ : On subordinate univalent functions, Trudy Mat. Inst. Steklov. **38** (1951), 68-71, Izdat. Akad Nauk, SSSR, Moscou.

37. ________ : On majorants of subordinate analytic functions I, Mat. Sb. N.S. **29** (71) (1951), 209-224.

38. ________ : On majorization of subordinate analytic functions II, ibid., 593-602.

39. ________ : Geometric theory of functions of a complex variable GITTL, Moscow (1952), Transl. Amer.Math. Soc. **26** (1969).

40. Goodman, A.W.: The rotation theorem for starlike univalent functions, Proc. Amer. Math. Soc. **4** (1953), 278-286.

41. ________ : Curvature under an analytic transformation, J. London Math. Soc. **43** (1968), 527-533.

42. ________ : Univalent Functions, Vol. I, Mariner, Tampa, Florida, 1983.

43. ________ : Univalent Functions, Vol. II, Mariner, Tampa, Florida, 1983.

44. ________ : Convex Functions of Bounded Type, Proc. Amer. Math. Soc. **92** (1984), 541-546.

45. ________ : Convex curves of Bounded Type, Internat. J. Math. Math. Sci. **8** (1985), 625-633.

46. ________ : More on convex functions of bounded type, Proc. Amer. Math. Soc. **97** (1986), 303-306.

47. Grownwall, T.: Some remarks on conformal representation, Ann. of Math. (2) **16** (1914-15), 72-76.

48. ________ : Sur la deformation dans la representation conforme, C.R. Acad. Sci. Paris **162** (1916), 249-252.

49. Gupta, V.P.; Jain, P.K.: Certain classes of univalent functions with negative coefficients, Bull. Austral. Math. Soc. **14** (1976), 409-416.

50. ________ : Certain classes of univalent functions with negative coefficients, ibid., **15** (1976), 467-473.

51. Hallenbeck, D.J.: Extreme points of classes of functions defined by subordination, Proc. Amer. Math. Soc. **46** (1974), 59-64.

52. Hallenbeck, D.J.; Mac Gregor, T.H.: Linear problems and convexity techniques in geometric function theory, Pitman Advance Publishing Program, Boston - London - Melbourne, 1984.

53. Hayman, W.K.: Schlichte Funktionen nach de Branges, Jahresber. Deutsch. Math. - Verein **89** (1987), 81-87.

54. Iliev G. Liubomir : Classical extremal problems in the theory of univalent functions, Proc. of the Sixteenth Spring Conference of the Union of Bulgarian Mathematicians, Sunny Beach, April 6-10, 1987, 9-34.

55. Jack, I.S. : Functions starlike and convex of order α, J. London Math. Soc. (2) **3** (1971), 469-474.

56. Janowski, W.: Extremal problems for a family of functions with positive real part and for some related families, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom Phys. **17** (1969), 633-637.

57. ____________ : Some extremal problems for certain families of analytic functions I, Ann. Polon. Math. **28** (1973), 297-326.

58. Jenkins, J.A., Univalent functions and conformal mappings, Springer Verlag, Berlin, (1965).

59. Juneja, O.P.; Shah, S.M. : Univalent functions with univalent Gelfond-Leontev derivatives, Bull. Austral. Math. Soc. **29** (1984), 329-348.

60. Kaplan, W.: Close-to-convex schlicht functions, Michigan Math. J. **1** (1952), 169-185.

61. Kapoor, G.P.;Mishra, A.K.: a^*-families of analytic functions, Internat. J. Math. Math. Sci. **7** (1984), 435-442.

62. Kasten, V.: On close-to-convex trinomials, Arch. Math. (Basel) **44** (1985), 65-72.

63. Kobori, A.: Uber Sternige und konvexe Abbildung, Mem. Coll. Sci. Kyoto A **15** (1932), 267-278.

64. Koebe, P.: Uber die Uniformisierung beliebiger analytischer Kurven, Nachr. Akad. Wiss.Gottingen Math.- Phys. Kl., 1907, 191-210.

65. Korickii, G.V.: On curvature of level lines and of their orthogonal trajectories in conformal mappings, Math. Sb. N.S. **37** (79) (1955), 103-116 (Russian).

66. Kossler, M.: Simple polynomials, Czechoslovak. Math. J. **1** (76) (1951), 5-15.

67. Koutroufiotis, D.: On Blaschke's Rolling Theorems, Arch. Math. (Basel) **23** (1972), 655-660.

68. Lakshma Reddy, G.; Padmanabhan, K.S.: p-valent regular functions with negative coefficients, Comment.Math. Univ. St. Paul. **32** (1983), 39-49.

69. Libera, R.J.: Univalent α-spiral functions, Canad. J. Math. **19** (1967), 449-456.

70. Loewner, K: Untersuchungen uber die Verzerrung bei Konformen Abbildungen des Einheitskreises $|z| < 1$, Leipzig Berichte **69**

71. Ma, W.C.; Mejia, D. and Minda, D.: Distortion theorems for Euclidean k-convex functions, Proc. Amer. Math. Soc. (to appear).

72. Mac Gregor, T.H.: Functions whose derivative has positive real part, Trans. Amer. Math. Soc. **104** (1962), 532-537.

73. ________ : A class of univalent functions, Proc. Amer. Math. Soc. **15** (1964), 311-317.

74. ________ : A subordination for convex functions of order α, J. London Math. Soc. **9** (1975), 530-536.

75. Martynov, Ju. A.: Uber geometrische Eigenschaften der Bogen der Niveaulinien bei schlichten konformen Abbildungen (Russian), Trudy Tomsk Cgosndarst. Univ. V.V. Kulbysev **210**, ser meh.-mat., Voprosy geom. Teor. Funkcil 6(1969) 53-61.

76. Marx, A. : Untersuchungen uber schlichte Abbildungen, Math. Ann. **107** (1932-33), 40-67.

77. McCarty, Carl, P. : Functions with real part greater than α, Proc. Amer. Math. Soc. **35** (1972), 211-216.

78. Mejia, D.; Minda, D.: Hyperbolic geometry in k-convex regions, Pacific J. Math. **141**, (1990), 333-354.

79. Merkes E.P. ; Robertson, M.S.; Scott, W.T.: On products of starlike functions, Proc. Amer. Math. Soc. **13** (1962), 960-964.

80. Milin, I.M.: Univalent functions and orthogonal systems, Izdat "Nauka", Moscow, 1971 (in Russian), English translation. Amer. Math. Soc. Providence, R.I., 1977.

81. Miller, S.S.; Mocanu, P.T.; Reade, M.O., All α-convex functions are univalent and starlike, Proc. Amer. Math. Soc. **37** (1973) 553-554.

82. Mirosnicenko, Ja.S.: Certain extremal problems of the theory of univalent functions (Russian), Izv. Vyssh. Uchebn. Zaved. Mat. (1965), No. 2 (45), 104-109.

83. ________ : On the question of the curvature of level curves (Russian), Trudy Tomsk.Gos. Univ. Ser. Meh.-Mat.**210** (1969), 62-65.

84. Mishra, A.K.; Sahu, P.: Convex hulls and extreme points of some families of multivalent functions, Indian J. Pure Appl. Math. **19** (1988), 865-874.

85. ________ : Extreme points of some families of analytic functions related to univalent functions, ibid., **20** (1989), 820-828.

86. Mitrinovic, D.S.: On the univalence of rational functions, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz. No. 634-No. 677 (1979), 221-227.

87. Mocanu, Petru, T.: Une propriete de convexite generalisee dans la theorie de la representation conforme, Mathematica (Cluj) **11** (34) (1969), 127-133.

88. Mogra, M.L.: A unified study of certain subclasses of univalent analytic functions, Ph.D. Thesis, IIT Kanpur, India (1976).

89. Nehari, Z.: Conformal mapping, Mc Graw Hill Book Co. Inc., (1952).

90. Nevanlinna, R.: Uber die schlichten Abbildungen des Einheitkreises. Finska Vetena Kaps-Soc. Forhandl (A), 62 (1920), No. 7, 145.

91. ________ : Uber die konforme Abbildung Strengebieten. Oeversikt av Finska-Vetenskaps Societeten Forhandlingar 63(A) No. 6 (1921).

92. Owa, S.; Aouf, M.K.: On subclasses of univalent functions with negative coefficients, Pure Appl. Math. Sci. **20** (1989), 131-139.

93. Owa, S.; Sekine, T.; Yaguchi, T.; Nunokawa, M.; Pashkouleva, D.: Notes on certain class of analytic functions with negative coefficients, Internat. J. Math. Math. Sci. **12** (1989), 199-204.

94. Ozaki, S.: On the theory of multivalent functions, Sci. Rep. Tokyo Burnika Daig. Sect. A (1941), 45-87.

95. Padmanabhan, K.S.: On certain classes of starlike functions in the unit disc, J. Indian Math. Soc. **32** (1968), 84-103.

96. Padmanabhan, K.S.; Parvatham, R.: On the univalence and convexity of partial sums of a certain class of analytic functions whose derivatives have positive real part, Indian J. Math. **16** (1974), 67-77.

97. Parvatham, R. and Shanmugam, T.N.: On analytic functions with reference to an integral operator, Bull. Austral. Math. Soc. **28** (1983), 207-215.

98. Patel, Jagannath : A study of univalence of Gelfond-Leontev derivatives of analytic functions and related function classes, Ph.D. Thesis, 1987, IIT, Kanpur, India.

99. Peschl, E. : Uber die Krummung von Niveaukurven bei der konformen Abbildung einfachzusammenhangender Gebiete auf das Innere eines Kreises, Math. Ann. **106** (1932), 574-594.

100. Pilat, Barbara: Sur une classe de fonctions normees univalentes dans le cercle unite, Ann. Univ. Mariae Curie-Sklodowska Sect. A **17** (1963), 63-73.

101. Pinchuk, B.: On starlike and convex functions of order α, Duke Math. J. **35** (1968), 721-734.

102. Pommerenke, Ch. : Univalent functions, Vandenhoeck and Ruprecht, Gottingen, 1975.

103. Prawitz, H.: Uber Mittelwerte analytischer Funktionen, Ark. Mat. Astron. och Fysik 20 (1927/28), No. 6, 1-12.

104. Rahman, Q.I. and Szynal, J.: On some classes of univalent polynomials, Canad. J.Math. **30** (1978), 332-349.

105. Rahman, Q.I. ; Waniurski, J.: Coefficient regions for univalent trinomials II, Ann. Univ. Mariae Curie-Sklodowska. Sect. A **32** (17) (1979) 189-219.

106. Raupach, E.: Eine Abschatzungsmethode fur die reellwertigen Losungen der Differentialgleichung $\Delta\alpha = -4/(1-z\bar{z})^2$, Bonner Math. Schriften, No. 9 (1960).

107. Reade, M.O.: Sur une classe de fonctions univalentes , C.R. Acad. Sci. Paris, **239** (1954), 1758-1759.

108. ________ : On close-to-convex univalent functions, Michigan Math. J. **3** (1955),59-62.

109. Reade, M.O.; Silverman, H.; Todorov, P.G. : On the starlikeness and convexity of a class of analytic functions, Rend. Circ. Mat. Palermo (2) **33** (1984), 265-272.

110. ________ : Classes of rational functions, Contemp. Math. **38** (1985), 99-103.

111. Robertson, M.S.: On the theory of univalent functions, Ann. of Math. **37** (1936), 374-408.

112. ________ : Univalent functions f(z) for which zf'(z) is spirallike, Michigan Math. J., **16** (1969), 97-101.

113. Rogosinski, W.: On subordinate functions, Proc. Cambridge Philos. Soc., **35** (1939), 1-26.

114. ________ : On the coefficients of subordinate functions, Proc. London Math. Soc., **48** (1943), 48-82.

115. Ruscheweyh, St.: New criteria for univalent functions, Proc. Amer. Math. Soc. **49**, (1975), 109-115.

116. ________ : Linear operators between classes of prestarlike functions, Comment. Math. Helv. **52** (1977), 497-509.

117. ________ : Extension of Szego's Theorem on the sections of univalent functions, SIAM J. Math. Anal. **19**, (1988), 1442-1449.

118. ________ : Sheil-Small, T.: Hadamard products of schlicht functions and the Polya Schoenberg Conjecture, Comment. Math. Helv. **48** (1973), 119-135.

119. Ruscheweyh, St.; Wirths, K.-J.: Uber die Koeffizienten spezieller schlichter polynome, Ann. Polon. Math. **28** (1973), 341-355.

120. Sakaguchi, K.: A note on p-valent functions, J. Math. Soc. Japan **14** (1962), 312-321.

121. Schild, A.: On a class of schlicht functions in the unit circle, Proc. Amer. Math. Soc. **5** (1954), 115-120.

122. Schober, G.: Univalent functions - selected topics, Lecture notes in Mathematics, **478**, Springer Verlag, Berlin - New York (1975).

123. ________ : Some conjectures for the class Σ, Topics in complex analysis, Proc. of the special session held in Fairland, Conn., Oct. 28-29, 1983, Contemp. Math. **38** (1985), 13-21.

124. Sekine, T.: On new generalized class of analytic functions with negative coefficients, Report of the Research Institute of Science and Technology, Nihon University, No. 32, (1987), 1-26.

125. Shaffer, D.B.: Distortion theorems for a special class of analytic functions, Proc. Amer. Math. Soc. **39** (1973), 281-287.

126. ________ : Epilogue, the Bieberbach conjecture, Topics in complex analysis, Proc. of the special session, held in Fairland, Conn., Oct. 28-29, 1983, Contemp. Math. **38** (1985), 139-141.

127. Shah Tao-Shing : Goluzin's number $(3-\sqrt{5})/2$ is the radius of superiority in subordination, Science Record, **1**, (1957), 25-28.

128. ________ : On the radius of superiority in subordination, ibid., 53-57.

129. Sheil-Small, T.; Silverman, H.; Silvia, E.M. : Convolution multipliers and starlike functions, J. Analyse Math. **41** (1982), 181-192.

130. Silverman, H. Univalent functions with negative coefficients, Proc. Amer. Math. Soc. **51** (1975), 109-116.

131. ________ : Extreme points of univalent functions with two fixed points, Trans. Amer. Math. Soc. **219** (1976), 387-395.

132. ________ : Subclasses of starlike functions, Rev. Roumaine Math. Pures Appl. **23** (1978), 1093-1099.

133. ________ : Univalent functions with varying arguments, Houston J. Math. **7** (1981), 283-287.

134. ________ : Order of starlikeness for multipliers of univalent functions, J. Math. Anal. Appl. **103**, (1984), 48-57.

135. ________ : Univalent functions having univalent derivatives, Rocky Mountain J. Math. **16**, (1986), 55-61.

136. ________ : Conjectures for univalent functions with negative coefficients, Int. Conference on new trends in Geom. Function Theory, July 26-29, 1990, Madras Univ., Madras, India.

137. Silverman, H.; Silvia, E.M.: Prestarlike functions with negative coefficients, Internat. J. Math. Math. Sci. **2** (1979), 427-439.

138. ________ : Fixed coefficient for subclasses of starlike functions, Houston J. Math. **7** (1981), 129-136.

139. ________ : Univalence preserving operators, Complex Variables Theory Appl. **5** (1986), 313-321.

140. Silvia, E.M. : On a subclass of spirallike functions, Proc. Amer. Math. Soc. **44** (1974), 411-420.

141. Singh, R. : On a class of starlike functions, Compositio Math. **19** (1967), 78-82; correction of an error in ibid., **21** (1969), 230-231.

142. Singh, R.; Singh, S. : Convex and starlike functions, Colloq. Math. **50** (1985), 127-135.

143. Singh, R.; Paul, S. : Linear sums of certain analytic functions, Proc. Amer. Math. Soc. **99**, (1987), 719-725.

144. Singh, V. : Some recent developments in the theory of univalent functions, Math. Student **55**, (1987), 7-26.

145. Spacek, L. : Contribution de la theorie des functions univalence Casop-pest. Math. Phys. **62** (1932), 12-19.

146. Srivastava, H.; Owa, S. : Certain classes of analytic functions with varying arguments, J. Math. Anal. Appl. **136**, (1988), 217-228.

147. Stroganoff, W. : Uber den arc f'(z) unter der Bedingung, dass f(z) die Konforme Abbildung eines sternartigen Gabietes auf das Innere des Einheitskreises der z-Ebene liefert, Trudy Mat. Inst. Steklov. **5** (1934), 247-258.

148. Strohhacker, E. : Beitrage zur Theorie der schlichten Funktionen, Math. Z., **37** (1933), 356-380.

149. Study, E. : Konforme Abbildung Einfachzusammenhangender Bereiche, B.C. Teubner, Leipzig and Berlin, 1913.

150. Styer, D. and Wright, D.J. : Convex functions with restricted curvature, Proc. Amer. Math. Soc., **109**, (1990), 981-990.

151. Suffridge, T.J. : Starlike functions as limits of polynomials, Advances in complex function theory, Lecture notes in Maths., **505**, Springer, Berlin-Heidelberg - New York, 1976, 164-202.

152. ________ : Polynomials in function theory, Topics in complex analysis, Proc. of the special session held in Fairland, Conn., Oct. 28-29, 1983, Contemp. Math. **38** (1985), 31-42.

153. Szego, G. : Zur theorie der schlichten Abbildungen, Math. Ann. **100** (1928), 188-201 (German version).

154. Taylor, A.E. : Introduction to Functional Analysis, John Wiley and Sons, New York, (1958).

155. Thout Reddy, R. : A study of certain subclasses of univalent analytic functions, Ph.D. Thesis, IIT Kanpur, India (1983).

156. Toeplitz, O. : Die linearen vollkommenen Raume der Funktionen theorie, Comment. Math. Helv. **23** (1949), 222-242.

157. Walsh, J.L. : Note on curvature on level curves of Green's function, Proc. Nat. Acad. Sci. **23** (1937), 84-89.

158. Wirths, K.-J. : Uber drehungssatze fur gewisse klassen holomorpher funktionen, Ber. Gesellsch. Math. Datenverarb. Bonn-Birlinghoven, Nr. 57, (1972), 71-76.

159. ________ : Coefficient bounds for convex functions of bounded type, Proc. Amer. Math. Soc. **103**, (1988), 525-530.

160. ________ : On a method to deal with convex functions of bounded type, Ann. Univ. Mariae Curie-Sklodowska Sect. A **41** (1987), 153-158.

161. Yamaguchi, K. : On functions satisfying Re(f(z)/z) > 0, Proc. Amer. Math. Soc. **17** (1966), 588-591.

162. Zmorovic, V.A. : On certain variational problems of the theory of univalent functions, Ukrain. Mat. Zurnal, **4** (1952), 276-298.

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